

THE STABLE FREE RANK OF SYMMETRY OF PRODUCTS OF SPHERES

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ABSTRACT. A well known conjecture in the theory of transformation groups states that if p is a prime and $(\mathbb{Z}/p)^r$ acts freely on a product of k spheres, then $r \leq k$. We prove this assertion if p is large compared to the dimension of the product of spheres. The argument builds on tame homotopy theory for non-simply connected spaces.

1. INTRODUCTION

Transformation group theory investigates symmetries of topological spaces. An important aspect of this program is to define and study invariants that distinguish spaces admitting lots of symmetries from less symmetric ones.

In this paper we concentrate on one of these invariants, the so called *free p -rank*

$$\max\{r \mid (\mathbb{Z}/p)^r \text{ acts freely on } X\},$$

defined for any topological space X and any prime number p . Here all groups are acting topologically. We recall the following fact from classical Smith theory.

Theorem 1.1. *The free p -rank of S^n is equal to*
$$\begin{cases} 1 & \text{for odd } n \\ 1 & \text{for even } n \text{ and } p = 2 \\ 0 & \text{for even } n \text{ and } p > 2. \end{cases}$$

In view of this theorem it is natural to look for a corresponding result, if X is not just a single sphere, but a product of spheres,

$$X = S^{n_1} \times S^{n_2} \times \dots \times S^{n_k}.$$

The following statement appears in several places in the literature either as a question [2, Question 7.2], [23, Problem 809] or as a conjecture [1, Conjecture 2.1], [3, Conjecture 3.1.4].

Conjecture 1.2. *If $(\mathbb{Z}/p)^r$ acts freely on X , then $r \leq k$.*

Actually, if p is odd, it is (in view of Theorem 1.1) reasonable to conjecture that r is bounded above by the number k_o of odd dimensional spheres in X . Conjecture 1.2, in this sharper form for odd p , has been verified in the following cases:

- 1) $k \leq 2$, see Heller [18]; $k \leq 3$, $p = 2$, see Carlsson [10].
- 2) $n_1 = \dots = n_k$ and in addition
 - a) the induced action on integral homology is trivial, see Carlsson [9], or
 - b) the induced action on integral homology is unrestricted, but if $p = 2$, then $n_i \neq 3, 7$, see Adem-Browder [2] (for $p \neq 2$ or $n_i \neq 1, 3, 7$), and Yalçın [32] (for $p = 2$ and $n_i = 1$).

- 3) assume $n_1 \leq n_2 \leq \dots \leq n_k$. Then $n_1 \geq 2$ and for all $1 \leq i \leq k-1$ either $n_i = n_{i+1}$ or $2n_i \leq n_{i+1}$. Furthermore, the induced $(\mathbb{Z}/p)^r$ -action on $\pi_*(X)$ is trivial and $p > 3 \dim X$. See Sørensen [28].

The following theorem is our main result. It settles a stable form of Conjecture 1.2.

Theorem 1.3. *If $p > 3 \dim X$ and $(\mathbb{Z}/p)^r$ acts freely on X , then $r \leq k_o$.*

Because $(\mathbb{Z}/p)^{k_o}$ acts freely on a product of k_o odd dimensional spheres, this result implies that the free p -rank of X is equal to k_o assumed that $p > 3 \dim X$. Theorem 1.3 also leads to the following estimate of the *free toral rank* of X which follows from [17, Theorem T].

Theorem 1.4. *If $(S^1)^r$ acts freely on X , then $r \leq k_o$.*

However, even for very large primes Theorem 1.3 cannot be deduced from this result by some sort of limiting process: Browder [8] constructed free actions of $(\mathbb{Z}/p)^r$ on $(S^m)^k$ for each odd $m \geq 3$ and $r \geq 4, k \geq r, p > km/2$ which are exotic in the sense that they cannot be extended to $(S^1)^r$ -actions.

Theorem 1.4 is implied by a theorem of Halperin [17] which yields upper bounds of free toral ranks in terms of homotopy Euler characteristics. The proof is based on rational homotopy theory applied to the Borel space $X_{(S^1)^r}$. Hence rational homotopy theory has become an important tool in the study of torus actions in general, see [4, Section 2] for a survey.

Because the rational homotopy type of $B(\mathbb{Z}/p)^r$ is trivial, it is obvious that in the context of Theorem 1.3, rational homotopy theory cannot be applied in a reasonable way to study the Borel space $X_{(\mathbb{Z}/p)^r}$. But *tame homotopy* theory seems a more promising approach. This theory was invented by Dwyer [14] and is modelled on Quillen's rational homotopy theory [25], but without immediately losing p -torsion information for all primes p . One of the first observations in tame homotopy theory may be phrased as follows: if X is an $(r-1)$ -connected space ($r \geq 1$) and $\pi_{r+k}(X)$ is a $\mathbb{Z}[p^{-1} \mid 2p-3 \leq k]$ -module for every $k \geq 1$, then the complexity of the Postnikov invariants of X should be comparable to that of a rational space due to the vanishing of the relevant higher reduced Steenrod power operations in the Postnikov pieces of X .

Nevertheless, in the original setup, Dwyer's theory could only be formulated for $r > 2$, see [14, 1.5]. In the tame setting, the restriction to simply connected spaces, which excludes the Borel spaces for $(\mathbb{Z}/p)^r$ -actions, seems unavoidable due to the non-vanishing of the p -fold Massey products $H^1(B\mathbb{Z}/p; \mathbb{F}_p) \rightarrow H^2(B\mathbb{Z}/p; \mathbb{F}_p)$ for any prime p , see [19, Theorem 14]. We will make more comments on this point later on.

To any space X one can associate the *Sullivan-de Rham algebra* [6, 31], a commutative graded differential algebra over \mathbb{Q} modelled on de Rham differential forms, which calculates the rational cohomology of X . This construction widens the scope of rational homotopy theory from simply connected to nilpotent spaces. A distinctive feature of this approach is the construction of a *minimal model* for any space X out of the Sullivan-de Rham algebra. The minimal model still calculates the rational cohomology of the underlying space, but in addition its associated vector space of indecomposables can be identified with the dual of $\pi_*(X) \otimes \mathbb{Q}$, if X is nilpotent, $H^*(X; \mathbb{Q})$ is finite dimensional in each degree and $\pi_1(X)$ is abelian.

For a proof of Theorem 1.4 one writes down a minimal model of the Borel space $X_{(S^1)^r}$ and argues that if $r > k_o$, then the cohomology of the minimal model is nonzero in arbitrarily high degrees. But this contradicts the fact that by the freeness of the action, $X_{(S^1)^r}$ is homotopy equivalent to the finite dimensional space $X/(S^1)^r$. The analysis of the minimal model that leads to these conclusions is carried out in the fundamental paper [17].

The generalization of the Sullivan-de Rham algebra to the tame setting is realized in the work of Cenkli and Porter [12] and is achieved by considering commutative graded differential algebras over \mathbb{Q} which are equipped with *filtrations* as a new structural ingredient. By definition, elements in filtration level q are divisible by any prime $p \leq q$, but not necessarily by larger primes, and filtration levels are additive under multiplication of elements. The construction of the Cenkli-Porter complex is based on differential forms similar to the Sullivan-de Rham algebra, but in order to keep sufficient control over filtration levels one works with differential forms defined on a cubical decomposition of the standard simplex. The *Cenkli-Porter theorem* states that the integration of forms yields a cochain map of the Cenkli-Porter complex in filtration level q to the singular cochain complex with $\mathbb{Z}[p^{-1} \mid p \leq q]$ -coefficients which in cohomology induces multiplicative isomorphisms. We will recall the essential steps of this construction in Section 3 of our paper. Our proof of the Cenkli-Porter theorem is different from the original approach in [12] in that we do not analyse the integration map, which is not multiplicative on the cochain level, but construct multiplicative cochain maps inducing isomorphisms in cohomology from the Cenkli-Porter and singular cochain complexes to a third cochain complex. This approach is inspired by the discussion of the de Rham theorem in [16, Theorem II.10.9] and enables us to take up and resolve the issue of possibly non-vanishing higher Massey products at the end of Section 3.

It is now desirable to replace the Cenkli-Porter complex by a smaller filtered commutative graded differential algebra which nicely reflects the homotopy type of the underlying space in a similar manner as the minimal algebra does in rational homotopy theory. In the tame setting, the most transparent and elementary approach to the construction of such an algebra is based on a *tame Hirsch lemma* which is used to build the desired small filtered cochain algebra along a Postnikov decomposition of the underlying space, compare the discussion in [13] for the rational case. For simply connected spaces such a result is proven in [26, p. 203]. The non-simply connected case, which is relevant for our purposes, is much more involved and was carried out in the remarkable diploma thesis of Sørensen [28]. After introducing the necessary notions we will state the tame Hirsch lemma in Section 4. The proof of this result, which is not available in the published literature, will be given in an appendix to our paper along the lines of [28]. I owe my thanks to Till Sørensen, who left academia some seventeen years ago, for producing a PDF scan of his work and to the FU Berlin for storing it on their preprint server.

With this machinery in hand we construct small approximative commutative \mathbb{F}_p -cochain models of Borel spaces associated to $(\mathbb{Z}/p)^r$ -spaces X under fairly general assumptions in Section 5. The main idea is to use a simultaneous Postnikov decomposition of the fibre and total space of the Borel fibration $X \hookrightarrow X_{(\mathbb{Z}/p)^r} \rightarrow B(\mathbb{Z}/p)^r$.

The cochain model resulting from this discussion is the starting point for our proof of Theorem 1.3 in Section 2. The argument is inspired in part by the paper [17], which provides techniques to simplify the analysis of free commutative graded differential algebras over fields of characteristic zero. We emphasize that as stated, some of the arguments in *loc. cit.* do not work over fields of prime characteristics. However, in the context of actions on products of spheres, with which we are mainly concerned, the resulting \mathbb{F}_p -cochain algebras are special enough so that the proof of Theorem 1.3 can be completed fairly quickly by a direct argument.

The main result of Section 5, Theorem 5.6, may be useful for other purposes in the cohomology theory of p -torus actions, which is now closely tied to the cohomology theory of $(S^1)^r$ -actions, if p is large. For example we hope that it is possible to generalize [17, Theorem T] in the sense that for large p , the free p -rank of a finite connected CW-complex with abelian fundamental group is

bounded above by the rational homotopy Euler characteristic of X . However, we will not discuss this in greater detail.

We believe that the methods developed in this paper are not sufficient to establish the general form of Conjecture 1.2 for small primes. This problem remains open.

Acknowledgements. I am grateful to Bill Browder for drawing my attention to [28], to Jonathan Bowden for proofreading a previous version of this manuscript and to the referees for numerous helpful comments and remarks.

2. PROOF OF THEOREM 1.3

Let p be a prime and assume $G := (\mathbb{Z}/p)^r$ acts freely on $X = S^{n_1} \times \dots \times S^{n_k}$. Denoting by k_e the number of even dimensional spheres in X , we assume that the dimensions n_1, \dots, n_{k_e} are even (and greater than 0) and n_{k_e+1}, \dots, n_k are odd. Because G acts freely, the Borel space $X_G := EG \times_G X$ is homotopy equivalent to the orbit space X/G . This is a topological manifold of dimension $\dim X$ and consequently $H^n(X_G; \mathbb{F}_p) = 0$ for all $n > \dim X$. Assume that $p > 3 \dim X$. In particular, p is odd.

Under these assumptions the following result will be proved at the end of Section 5.

Proposition 2.1. *There are free associative, commutative graded differential algebras with unit (E^*, d_E) and (M^*, d_M) over \mathbb{F}_p with the following properties:*

- 1) *as graded algebras $E^* = \mathbb{F}_p[t_1, \dots, t_r] \otimes \Lambda(s_1, \dots, s_r) \otimes M^*$, where each t_i is of degree 2 and each s_i is of degree 1.*
- 2) *the differential d_E is zero on $\mathbb{F}_p[t_1, \dots, t_r] \otimes \Lambda(s_1, \dots, s_r) \otimes 1$ and the map $E^* \rightarrow M^*$, $t_i, s_i \mapsto 0$, is a cochain map.*
- 3) *M^* is the tensor product of commutative graded differential algebras (M_j^*, d_{M_j}) , $1 \leq j \leq k$, that are defined as follows:*
 - a) *$M_j^* = \Lambda(\sigma_j)$ with $\deg(\sigma_j) = n_j$ and $d_{M_j}(\sigma_j) = 0$, if $j > k_e$ (i.e. if n_j is odd),*
 - b) *$M_j^* = \mathbb{F}_p[\tau_j]$ with $\deg(\tau_j) = n_j$ and $d_{M_j}(\tau_j) = 0$, if $j \leq k_e$ and $2n_j - 1 > \dim X + 1$,*
 - c) *$M_j^* = \mathbb{F}_p[\tau_j] \otimes \Lambda(\eta_j)$ with $\deg(\tau_j) = n_j$, $\deg(\eta_j) = 2n_j - 1$ and $d_M(\tau_j) = 0$, $d_M(\eta_j) = \tau_j^2$, if $j \leq k_e$ and $2n_j - 1 \leq \dim X + 1$,*
- 4) *each monomial in t_1, \dots, t_r of cohomological degree at least $\dim X + 1$ represents the zero class in $H^*(E)$.*

The short exact sequence of \mathbb{F}_p -cochain algebras

$$\mathbb{F}_p[t_1, \dots, t_r] \otimes \Lambda(s_1, \dots, s_r) \rightarrow E^* \rightarrow M^*$$

models the Borel fibration $X \hookrightarrow X_G \rightarrow BG$ up to degree $\dim X + 1$ (respectively up to degree $\dim X + 2$ as far as the monomials in t_1, \dots, t_r are concerned) implying the last statement of Proposition 2.1. In rational homotopy theory the Grivel-Halperin-Thomas theorem [4, Theorem 2.5.1] yields a similar short exact sequence of rational differential graded algebras for spaces equipped with $(S^1)^r$ -actions that models the Borel fibration in all degrees. In this case $\mathbb{F}_p[t_1, \dots, t_r] \otimes \Lambda(s_1, \dots, s_r)$ is replaced by a polynomial algebra $\mathbb{Q}[t_1, \dots, t_r]$, the minimal model of $B(S^1)^r$, and the algebra M^* can be chosen as the minimal model of X . If X is a product of spheres as before, this minimal model looks very similar to the algebra M^* appearing in Proposition 2.1. In Section 5 tame homotopy theory will be used to establish a counterpart to the Grivel-Halperin-Thomas theorem for G -spaces and cohomology with \mathbb{F}_p -coefficients which models the Borel fibration up to degree $\dim X + 1$ and leads to the statement of Proposition 2.1. In the description of M^* the two cases $2n_j - 1 > \dim X + 1$ and $2n_j - 1 \leq \dim X + 1$ need to be

distinguished, because for $p > 3 \dim X$ the procedure of Section 5 constructs free generators of the algebra E^* only up to degree $\dim X + 1$. When the algebra E^* has been simplified, we can easily add generators η_j of degrees $2n_j - 1 > \dim X + 1$, see below.

Starting from (E^*, d_E) we shall construct a commutative graded differential algebra (F^*, δ_F) that is easier to handle. Let

$$F^* := \mathbb{F}_p[t_1, \dots, t_r] \otimes M^*$$

be obtained from E^* by dividing out the ideal (s_1, \dots, s_r) . The induced differential on F^* is denoted by d_F .

Inspired by the construction of *pure towers* in [17], we deform d_F to another differential δ_F on F^* as follows: δ_F is a derivation that vanishes on $\mathbb{F}_p[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]$ and satisfies

$$\begin{aligned} \delta_F(\sigma_j) &= \pi(d_F(\sigma_j)) \\ \delta_F(\eta_j) &= \pi(d_F(\eta_j)). \end{aligned}$$

where $\pi : F^* \rightarrow F^*$ is the projection onto $\mathbb{F}_p[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]$ given by evaluating the odd degree generators σ_j, η_j at 0. It is easy to verify that $\delta_F^2 = 0$.

For $l \geq 0$ let $\Sigma^l \subset F^*$ be the $\mathbb{F}_p[t_1, \dots, t_r]$ -linear subspace generated by the monomials in M^* containing exactly l of the odd degree generators σ_j, η_j , in particular $\Sigma^l = 0$ for $l > k$ by the anticommutativity of the product (recall that p is odd). We set $\Sigma^+ := \bigoplus_{l \geq 1} \Sigma^l$. This is a nilpotent ideal in F^* .

Lemma 2.2. *For all $l \geq 1$ the differential δ_F maps Σ^l to Σ^{l-1} . Furthermore, the image of $\delta_F - d_F$ is contained in Σ^+ .*

Proof. The first assertion holds by the definition and derivation property of δ_F .

The second assertion holds for the generators σ_j and η_j , because $\text{im}(\text{id} - \pi) \subset \Sigma^+$, it holds for the generators t_i , because δ_F and d_F send these elements to zero and it holds for the generators τ_j , because each $d_F(\tau_j)$ is of odd degree and therefore contained in Σ^+ . This implies the second assertion in general, since Σ^+ is an ideal in F^* and $\delta_F - d_F$ is a derivation. \square

The elements t_i , $1 \leq i \leq r$, and τ_j , $1 \leq j \leq k_e$, represent cocycles in (F^*, δ_F) . Let $[t_i]$ and $[\tau_j]$ be the corresponding cohomology classes. The proof of Theorem 1.3 depends on the following crucial fact.

Proposition 2.3. *The elements $[t_i]$ are nilpotent in $H^*(F, \delta_F)$. The same holds for $[\tau_j]$, if $2n_j - 1 \leq \dim X + 1$.*

Proof. As noted earlier each monomial in t_1, \dots, t_r of cohomological degree at least $\dim X + 1$ represents the zero class in $H^*(E)$. Let m be such a monomial and write $m = d_E(c)$ for a cochain $c \in E^*$. We will show that m is a coboundary in (F^*, δ_F) as well.

First, after dividing out the ideal (s_1, \dots, s_r) , we get an analogous equation $m = d_F(c)$ for some $c \in F^*$. By Lemma 2.2 we conclude $\delta_F(c) = m + \omega$ where $\omega \in \Sigma^+$. Let c_1 be the component of c in Σ^1 . Lemma 2.2 and the fact that $m \in \Sigma^0$ imply the equation $\delta_F(c_1) = m$. This shows that m is a coboundary in (F^*, δ_F) .

In particular we have shown that the classes $[t_i] \in H^*(F, \delta_F)$ are nilpotent.

The cochain algebra (F^*, δ_F) has a decreasing filtration given by

$$\mathcal{F}_\gamma(F^*) = \mathbb{F}_p[t_1, \dots, t_r]^{\geq \gamma} \otimes M^*$$

where $\gamma \in \mathbb{N}$ denotes the cohomological degree. Our previous argument and the fact that each τ_j is a cocycle in (F^*, δ_F) imply that each element in $\Sigma^0 \subset F^*$ in filtration level at least $\dim X + 1$ is a coboundary in (F^*, δ_F) .

Now pick a $j \in \{1, \dots, k_e\}$ with $2n_j - 1 \leq \dim X + 1$. Recall (see the description of E^* in Proposition 2.1) that

$$d_F(\eta_j) = \tau_j^2 \mod \mathcal{F}_2(F).$$

By the definition of δ_F we obtain

$$\delta_F(\eta_j) = \pi(\tau_j^2) = \tau_j^2 \mod \mathcal{F}_2(F),$$

since the map π preserves the ideal $(t_1, \dots, t_r) = \mathcal{F}_2(F)$. This implies that τ_j^2 is δ_F -cohomologous to a cocycle $c \in \mathcal{F}_2(F^*, \delta_F)$. Hence, $(\tau_j^2)^{\dim X}$ is δ_F -cohomologous to $c^{\dim X} \in \mathcal{F}_{2\dim X}(F)$. We can split $c^{\dim X}$ into a sum $c_0 + c^+$ where $c_0 \in \Sigma^0 \cap \mathcal{F}_{2\dim X}(F)$ and $c^+ \in \Sigma^+ \cap \mathcal{F}_{2\dim X}(F)$. As noted earlier, c_0 is δ_F -cohomologous to zero. Because Σ^+ is nilpotent, the element c^+ is nilpotent.

We conclude that $\tau_j^{2\dim X}$ is δ_F -cohomologous to a nilpotent cocycle in (F^*, δ_F) . \square

After these preparations we can finish the proof of Theorem 1.3. At first we adjoin new exterior generators η_j of degree $2n_j - 1$ to F^* for all $j \in \{1, \dots, k_e\}$ satisfying $2n_j - 1 > \dim X + 1$. The differential δ_F is extended to a differential on this new graded commutative algebra by setting $\delta_F(\eta_j) := \tau_j^2$, if $2n_j - 1 > \dim X + 1$. This is possible, because $\delta_F(\tau_j) = 0$ by construction of δ_F . This new commutative graded differential algebra is still denoted by (F^*, δ_F) . Together with Proposition 2.3 we see that the elements t_i , $1 \leq i \leq r$, and τ_j , $1 \leq j \leq k_e$, define nilpotent classes in $H^*(F, \delta_F)$. This implies that $H^*(F, \delta_F)$ is a finite dimensional \mathbb{F}_p -vector space.

We consider the ideal

$$I = (\delta_F(\eta_1), \dots, \delta_F(\eta_{k_e}), \delta_F(\sigma_{k_e+1}), \dots, \delta_F(\sigma_k)) \subset \mathbb{F}_p[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]$$

contained in $\text{im}(\delta_F)$ and obtain an inclusion

$$\mathbb{F}_p[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]/I \subset H^*(F, \delta_F).$$

Here we use the fact that the coboundaries in (F^*, δ_F) are contained in the ideal $I \cdot F^*$, whose intersection with $\mathbb{F}_p[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]$ is equal to I . This can be seen by applying the evaluation map π . We conclude that $\mathbb{F}_p[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]/I$ is a finite dimensional \mathbb{F}_p -vector space. Because I is generated by homogenous elements of positive degree, it does not contain a unit of $\mathbb{F}_p[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]$ and hence there is a minimal prime ideal $\mathfrak{p} \subset \mathbb{F}_p[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]$ containing I . The quotient $\mathbb{F}_p[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]/\mathfrak{p}$ is both a finite dimensional \mathbb{F}_p -vector space and an integral domain. Hence $\mathfrak{p} = (t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e})$ and consequently $\text{height}(\mathfrak{p}) = r + k_e$. By Krull's Principal Ideal Theorem, see [15, Theorem 10.2], the number of generators of I must be at least $r + k_e$. From the definition of I we derive the inequality $k_e + k_o \geq r + k_e$. This implies $k_o \geq r$ and finishes the proof of Theorem 1.3.

3. TAME HOMOTOPY THEORY VIA DIFFERENTIAL FORMS

The remainder of our paper is devoted to the construction of small approximative \mathbb{F}_p -cochain models of Borel spaces associated to $(\mathbb{Z}/p)^r$ -spaces in Theorem 5.6. Then the corresponding cochain models for $(\mathbb{Z}/p)^r$ -actions on products of spheres will lead directly to the cochain algebras (E^*, d_E) and (M^*, d_M) appearing in Proposition 2.1.

First, in this section, we collect some fundamental notions and constructions from tame homotopy theory via polynomial forms as initiated in [12] and further developed in [5, 21, 24, 26, 28, 30].

As in the rational situation [6] it is convenient to formulate the theory in the language of simplicial sets. The reader unfamiliar with this may consult the introductory text [22].

Prime numbers are henceforth denoted by the letter l . For $q \in \mathbb{N}$ we set

$$\mathbb{Q}_q := \mathbb{Z}[l^{-1} \mid l \leq q],$$

i.e. \mathbb{Q}_q is the smallest subring of \mathbb{Q} where all primes $l \leq q$ are invertible. In particular, $\mathbb{Q}_0 = \mathbb{Q}_1 = \mathbb{Z}$.

Definition 3.1. A *filtered cochain complex* is a cochain complex (V^*, d) over \mathbb{Q} , graded over \mathbb{N} , together with an increasing sequence of subcomplexes

$$V^{*,0} \subset V^{*,1} \subset V^{*,2} \subset \dots \subset V^*$$

so that $V^{*,q}$ is a cochain complex of \mathbb{Q}_q -modules. We call $V^{*,q}$ the subcomplex of *filtration level* q . A *filtered graded differential algebra* is a filtered cochain complex (A^*, d) which is equipped with an associative multiplication with unit $1 \in A^{0,0}$, which restricts to maps $A^{*,q} \otimes A^{*,q'} \rightarrow A^{*,q+q'}$. Furthermore, d is assumed to act as a graded derivation. If the multiplication is graded commutative, then A^* is called a *filtered commutative graded differential algebra* or *filtered CGDA*.

If A^* is a filtered cochain complex or filtered graded differential algebra and $x \in A^p$, we call p the *degree* of x and $\min\{q \mid x \in A^{p,q}\}$ the *filtration degree* of x . The category of filtered CGDAs and filtration preserving CGDA maps is denoted by \mathcal{A} . It can be equipped with the structure of a closed model category [21, 25, 26, 28, 30], but this fact will not be needed for our discussion.

Let V_1 and V_2 be two filtered cochain complexes. Their *tensor product* is the cochain complex $V_1 \otimes V_2$ with the usual grading and differential together with the filtration

$$(V_1 \otimes V_2)^q := \sum_{q_1+q_2=q} V_1^{*,q_1} \otimes V_2^{*,q_2} \otimes \mathbb{Q}_q \subset V_1^* \otimes V_2^*,$$

where \sum denotes a sum which need not be direct. The same definition applies to filtered CGDAs. The tensor product defines the coproduct in \mathcal{A} .

If V is a filtered cochain complex, let ΛV denote the free CGDA generated by V . It is equipped with a filtration as described in the preceding paragraph. Note that this is an algebra with unit $1 \in (\Lambda V)^{0,0}$ and that the filtration degree of a monomial $v_1 \cdot \dots \cdot v_n$, $v_i \in V$, is less than or equal to the sum of the filtration degrees of the v_i .

Sometimes the following variation of the tensor product will be useful. The *filtrationwise tensor product* $V_1 \overline{\otimes} V_2$ is defined as the tensor product complex $V_1 \otimes V_2$, but equipped with the filtration

$$(V_1 \overline{\otimes} V_2)^{*,q} := V_1^{*,q} \otimes V_2^{*,q}.$$

The prototypical example of a filtered CGDA is provided by the cubical cochain algebra of Cenkli and Porter [12] which refines the Sullivan-de Rham algebra [6, §1] used in rational homotopy theory. Whereas the latter is based on forms on the standard simplices $\Delta^n \subset \mathbb{R}^{n+1}$, the Cenkli-Porter construction works with forms on a cubical decomposition of Δ^n . This is essential for defining a filtration on the resulting cochain algebra in such a way that in each positive filtration level one gets a cohomology theory in the sense of Cartan [11]. We recall the essential steps of this construction.

We cubically decompose the standard n -simplex by regarding it as the subset

$$\Delta^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid 0 \leq x_i \leq 1, \prod_{i=0}^n x_i = 0\}$$

of the boundary of the $(n + 1)$ -dimensional cube. The vertices v_0, \dots, v_n of Δ^n are given by

$$v_i := (1, 1, \dots, 1, 0, 1, \dots, 1)$$

with 0 located at the i -th entry, $0 \leq i \leq n$. The inclusion of the i -th face into Δ^n and the i -th collapse onto Δ^n ($0 \leq i \leq n$) are the maps

$$\begin{aligned} \delta_i : \Delta^{n-1} &\rightarrow \Delta^n, & (x_0, \dots, x_{n-1}) &\mapsto (x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_{n-1}) \\ \sigma_i : \Delta^{n+1} &\rightarrow \Delta^n, & (x_0, \dots, x_{n+1}) &\mapsto (x_0, \dots, x_{i-1}, x_i \cdot x_{i+1}, x_{i+2}, \dots, x_{n+1}). \end{aligned}$$

We consider the free commutative graded algebra over \mathbb{Z}

$$\mathbb{Z}[t_0, \dots, t_n] \otimes \Lambda_{\mathbb{Z}}(dt_0, \dots, dt_n)$$

with generators t_0, \dots, t_n in degree 0 and dt_1, \dots, dt_n in degree 1. To a monomial

$$t_0^{\alpha_0} dt_0^{\epsilon_0} \cdot \dots \cdot t_n^{\alpha_n} dt_n^{\epsilon_n}, \quad \alpha_i \geq 0, \quad 0 \leq \epsilon_i \leq 1,$$

in this algebra we assign the *filtration degree* $\max_i \{\alpha_i + \epsilon_i\}$. The monomials of filtration degree 0 are exactly the constant ones. Let I be the homogenous ideal generated by the monomials with $\alpha_i + \epsilon_i > 0$ for all i . The quotient

$$\mathbb{Z}[t_0, \dots, t_n] \otimes \Lambda_{\mathbb{Z}}(dt_0, \dots, dt_n) / I$$

is again a graded algebra and is called the *algebra of compatible polynomial forms on the cubical decomposition of Δ^n* .

For $p, q \geq 0$, we define

$$T_n^{p,q}(\mathbb{Z}) \subset (\mathbb{Z}[t_0, \dots, t_n] \otimes \Lambda_{\mathbb{Z}}(dt_0, \dots, dt_n) / I)^p$$

as the \mathbb{Z} -module generated by the (cosets of) monomials of degree p and filtration degree at most q . Exterior product of polynomial forms induces maps

$$\wedge : T_n^{p_1, q_1}(\mathbb{Z}) \otimes T_n^{p_2, q_2}(\mathbb{Z}) \rightarrow T_n^{p_1 + p_2, q_1 + q_2}(\mathbb{Z}).$$

We define a coboundary map

$$d : T_n^{p,q}(\mathbb{Z}) \rightarrow T_n^{p+1,q}(\mathbb{Z})$$

by $x_i \mapsto dx_i$, $dx_i \mapsto 0$ and extend it to the whole of $T_n^{p,q}(\mathbb{Z})$ by the Leibniz rule. Upon defining

$$T_n^{p,q} := T_n^{p,q}(\mathbb{Z}) \otimes \mathbb{Q}_q$$

we obtain a filtered CGDA T_n^* for each $n \geq 0$. Via pullback of forms, the maps δ_i and σ_i induce maps of filtered CGDAs

$$\begin{aligned} \partial_i : T_n^* &\rightarrow T_{n-1}^* \\ s_i : T_n^* &\rightarrow T_{n+1}^* \end{aligned}$$

that satisfy the simplicial identities. In other words, $(T_n^*)_{n \in \mathbb{N}}$ is a simplicial filtered CGDA.

Let \mathcal{S} denote the category of simplicial sets. For any simplicial set $X \in \mathcal{S}$, we define a filtered CGDA

$$T^{*,q}(X) := \text{Mor}_{\mathcal{S}}(X, T^{*,q}),$$

which is called the *Cenkl-Porter complex of polynomial forms on X* . Note that by definition $T^{*,0}(X)$ is concentrated in degree 0 and can be identified with $\text{Map}(\pi_0(X), \mathbb{Z})$. Conversely, if $A \in \mathcal{A}$ is a filtered CGDA, its *simplicial realization* is the simplicial set $\|A\|$ defined by

$$\|A\|_n := \text{Mor}_{\mathcal{A}}(A, T_n^*)$$

for $n \geq 0$. Note that corresponding constructions exist in rational homotopy theory, see [6].

For a proof of the following almost tautological statement, see [6, 8.1].

Proposition 3.2. *Let X be a simplicial set and A be a filtered CGDA. Then there is a canonical bijection*

$$\mathrm{Mor}_S(X, \|A\|) \approx \mathrm{Mor}_A(A, T(X))$$

which is natural in A and X . In other words the simplicial realization $\| - \|$ and the Cenkli-Porter functor $T(-)$ define a pair of right adjoint contravariant functors.

Let us fix the notation $\Psi_A : A \mapsto T(\|A\|)$ for the unit defined by this adjunction.

We will now compare $H^*(T^{*,q}(X))$ with the simplicial cohomology of X . This can be done most in the axiomatic setting described in [11]. We call a simplicial cochain complex $(A_n^*)_{n \in \mathbb{N}}$ over a commutative ring R with identity a *cohomology theory*, if the sequence of simplicial R -modules

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \rightarrow \dots$$

is exact, if the kernel ZA^0 of the differential $d : A^0 \rightarrow A^1$ is simplicially trivial, i.e. all face and degeneracy maps are isomorphisms, and if for each $p \geq 0$, the simplicial abelian group A^p is contractible, cf. [11, Section 2]. Concerning the latter property, recall [22, Theorem 22.1] that for a simplicial abelian group $(G_n)_{n \in \mathbb{N}}$, the p -th homotopy group $\pi_p(G)$ can be computed as the p -th homology of the chain complex

$$\dots \rightarrow G_n \rightarrow G_{n-1} \rightarrow G_{n-2} \rightarrow \dots$$

with differentials equal to the alternating sum of the face operators of G . Given a cohomology theory A^* , we call the R -module $R(A) := (ZA^0)_0$ the *coefficients* of A . Furthermore we get an R -cochain complex

$$A^*(X) := \mathrm{Mor}_S(X, A^*)$$

whose cohomology is naturally isomorphic to $H^*(X; R(A))$, the simplicial cohomology of X with coefficients in $R(A)$, see [11, Théorème 2.1].

Proposition 3.3. *For each $q \geq 1$, the simplicial CGDA $T^{*,q}$ defines a cohomology theory with coefficients \mathbb{Q}_q .*

Proof. The sequence

$$T^{0,q} \xrightarrow{d} T^{1,q} \xrightarrow{d} \dots$$

of simplicial \mathbb{Q}_q -modules is exact by the cubical Poincaré lemma [12, Proposition 3.5]. The contractibility of the simplicial sets $T^{p,q}$ for all $p \geq 0$ follows from the cubical extension lemma [12, Lemma 3.1]. That the kernel of the differential $T^{0,q} \rightarrow T^{1,q}$ is simplicially trivial is obvious (it consists of constant monomials). \square

It is exactly at this point, where division by numbers smaller than or equal to q is required: for $2 \leq k \leq q$ the closed form $t^{k-1}dt$ lives in filtration level q and is the coboundary of t^k/k . Furthermore, notice that $T^{*,0}$ is not a cohomology theory. Indeed $T^{0,0} \rightarrow T^{1,0} \rightarrow T^{2,0} \rightarrow \dots$ is exact and $T^{0,0}$ is simplicially trivial, but $T^{0,0}$ is not connected.

The last proposition implies that if $q \geq 1$ and M is a \mathbb{Q}_q -module, then $M \otimes T^{*,q}$ is a cohomology theory with coefficients M . This follows from the universal coefficient theorem and the fact that $T^{*,q}$ consists of torsion free and hence flat \mathbb{Q}_q -modules, cf. [29, Theorem 5.3.14].

Corollary 3.4. *Let $q \geq 1$ and let M be a \mathbb{Q}_q -module. Then, for any $p \geq 0$, the simplicial \mathbb{Q}_q -module $M \otimes T^{p,q}$ is contractible and for any $p \geq 1$, the simplicial \mathbb{Q}_q -module $M \otimes ZT^{p,q}$ is an Eilenberg-MacLane complex of type (M, p) .*

Proof. Both statements are implied by the fact that $M \otimes T^{*,q}$ is a cohomology theory. The first statement is then immediate. The second statement is proved by an inductive argument as in [11, Théorème 1] combined with the equation $M \otimes ZT^{p,q} = \ker(\text{id} \otimes d) : M \otimes T^{p,q} \rightarrow M \otimes T^{p+1,q}$ which follows from the universal coefficient theorem applied to the cochain complex of flat \mathbb{Q}_q -modules $0 \rightarrow T^{p,q} \rightarrow T^{p+1,q} \rightarrow T^{p+2,q} \rightarrow \dots$ \square

In order to study the relation of $H^*(T^{*,q}(X))$ and $H^*(X; \mathbb{Q}_q)$, we first introduce for each $q \geq 1$ the cohomology theory $(C_n^{*,q})_{n \in \mathbb{N}}$, cf. [11, Section 3], with coefficients \mathbb{Q}_q , where $C_n^{*,q}$ is the simplicial cochain complex $C^*(\Delta[n]; \mathbb{Q}_q)$ of the simplicial n -simplex. For each simplicial set X , we will identify $C^{*,q}(X)$ with $C^*(X; \mathbb{Q}_q)$, the simplicial cochain complex of X with coefficients in \mathbb{Q}_q .

For $q \geq 1$ we introduce a third simplicial cochain complex $((T \overline{\otimes} C)_n^{*,q})_{n \in \mathbb{N}}$ over \mathbb{Q}_q with $(T \overline{\otimes} C)_n^{*,q} := T_n^{*,q} \otimes C_n^{*,q}$ and with face and degeneracy maps which are the tensor products of the corresponding maps on $T_n^{*,q}$ and $C_n^{*,q}$. The Künneth formula [29, Corollary 5.3.4] and the Eilenberg-Zilber theorem [20, Theorem 8.1] imply that $(T \overline{\otimes} C)^{*,q}$ is again a cohomology theory with coefficients \mathbb{Q}_q . We have canonical maps $T^{*,q} \rightarrow (T \overline{\otimes} C)^{*,q}$ and $C^{*,q} \rightarrow (T \overline{\otimes} C)^{*,q}$ of simplicial cochain complexes given by

$$\begin{aligned} T_n^{*,q} &= T_n^{*,q} \otimes (\mathbb{Q}_q \cdot 1) \hookrightarrow T_n^{*,q} \otimes C_n^{*,q} \\ C_n^{*,q} &= (\mathbb{Q}_q \cdot 1) \otimes C_n^{*,q} \hookrightarrow T_n^{*,q} \otimes C_n^{*,q}. \end{aligned}$$

These induce isomorphisms of coefficients and hence [11, Proposition 2] we get induced isomorphisms

$$\begin{aligned} H^*(T^{*,q}(X)) &\cong H^*((T \overline{\otimes} C)^{*,q}(X)) \\ H^*(C^{*,q}(X)) &\cong H^*((T \overline{\otimes} C)^{*,q}(X)), \end{aligned}$$

which are natural in X , for each $q \geq 1$. Now we observe that exterior multiplication of cubical forms and the cup product of simplicial cochains define maps of simplicial cochain complexes

$$\begin{aligned} T_n^{*,q_1} \otimes T_n^{*,q_2} &\rightarrow T_n^{*,q_1+q_2} \\ C_n^{*,q_1} \otimes C_n^{*,q_2} &\rightarrow C_n^{*,q_1+q_2} \\ (T \overline{\otimes} C)_n^{*,q_1} \otimes (T \overline{\otimes} C)_n^{*,q_2} &\rightarrow (T \overline{\otimes} C)_n^{*,q_1+q_2} \end{aligned}$$

which are compatible with the simplicial cochain maps $T_n^{*,q} \rightarrow (T \overline{\otimes} C)_n^{*,q}$ and $C_n^{*,q} \rightarrow (T \overline{\otimes} C)_n^{*,q}$ considered before. We hence obtain

Theorem 3.5 (Cenkl-Porter theorem [12]). *For $q \geq 1$ there are isomorphisms of \mathbb{Q}_q -modules*

$$H^*(T^{*,q}(X)) \cong H^*(X; \mathbb{Q}_q)$$

which are natural in X . These are compatible with the multiplicative structures in the sense that

$$\begin{array}{ccc} H^{p_1}(T^{*,q_1}(X)) \otimes H^{p_2}(T^{*,q_2}(X)) & \xrightarrow{\wedge} & H^{p_1+p_2}(T^{*,q_1+q_2}(X)) \\ \downarrow \cong & & \downarrow \cong \\ H^{p_1}(X; \mathbb{Q}_{q_1}) \otimes H^{p_2}(X; \mathbb{Q}_{q_2}) & \xrightarrow{\cup} & H^{p_1+p_2}(X; \mathbb{Q}_{q_1+q_2}) \end{array}$$

commutes for all $q_1, q_2 \geq 1$ and multiplicative units are mapped to each other.

Recall that the isomorphism $H^*(T^{*,q}(X)) \cong H^*(X; \mathbb{Q}_q)$ is induced by the zig-zag sequence of filtered graded differential algebras

$$(3.6) \quad T^*(X) \rightarrow (T \overline{\otimes} C)^*(X) \leftarrow C^*(X)$$

where both arrows induce isomorphisms in cohomology. In contrast, the proof of Theorem 3.5 in [12] is based on a single map of filtered cochain complexes

$$\int : T^{*,q}(X) \rightarrow C^*(X, \mathbb{Q}_q)$$

induced by integration of forms. One can indeed show that \int induces an isomorphism of cohomology groups which is compatible with multiplicative structures. However, \int is not multiplicative on the cochain level. Using the uniqueness of natural transformations of cohomology theories [11, Proposition 2] one can show that the isomorphisms induced by \int and the one in our Theorem 3.5 coincide.

Concerning filtration level zero, we note the existence of a canonical isomorphism

$$H^*(T^{*,0}(X)) \cong H^0(X; \mathbb{Z}).$$

Using this the multiplicativity property in Theorem 3.5 holds for $q_1, q_2 \geq 0$.

As an addendum to the Cenkli-Porter theorem, we remark that for any $q \geq 1$ and for any prime $l > q$, we have isomorphisms

$$H^*(T^{*,q}(X) \otimes \mathbb{F}_l) \cong H^*(X; \mathbb{F}_l),$$

which are multiplicative for filtration levels $q_1, q_2 \geq 1$ with $l > q_1 + q_2$. This is shown by tensoring the cochain complexes appearing in (3.6) with \mathbb{F}_l and applying the universal coefficient theorem.

We finish this section with a discussion of higher Massey products. Let l be an odd prime. It is well known [19] that the l -fold Massey product $H^1(B\mathbb{Z}/l; \mathbb{F}_l) \rightarrow H^2(B\mathbb{Z}/l; \mathbb{F}_l)$ is nonzero. Because squares of elements of degree one in commutative cochain algebras are equal to zero, one might wonder if this nonvanishing result does not contradict the Cenkli-Porter theorem and in particular the addendum stated above. The following discussion will clarify this point.

First we recall that Massey products are defined on singular cochain algebras $C^*(X; R)$ of topological spaces X , respectively simplicial cochain complexes of simplicial sets X , where R is some commutative ring with unit. Because our proof of Theorem 3.5 is based on (3.6) in which all maps are multiplicative on the cochain level, we can use the filtered cochain algebra $T^{*,*}(X)$ or - in view of the addendum to the Cenkli-Porter theorem - the complexes $T^{*,q}(X) \otimes \mathbb{F}_l$ to construct higher Massey products in $H^*(X; \mathbb{F}_l)$. But now the filtration structure of $T^{*,*}(X)$ has to be considered. More precisely, when constructing an l -fold Massey product of a cochain $c \in T^{1,1}(X) \otimes \mathbb{F}_l$, we have to perform $l - 1$ multiplications (cf. the construction of Massey products in [19]) and hence arrive at elements in $T^{*,l}(X) \otimes \mathbb{F}_l$. But this group is zero, because l is invertible in $T^{*,l}(X)$.

This reasoning indicates that it is exactly the filtration structure together with its divisibility properties that allows us to generalize the construction of the Sullivan-de Rham algebra to a non-rational setting.

4. TAME HIRSCH LEMMA

In this section we shall explain a technique that will enable us to replace the Cenkli-Porter complex by a smaller filtered CGDA which reflects the topological structure of the underlying simplicial set in a similar manner as the minimal model does in rational homotopy theory.

Of fundamental importance is the *filtration function* $\bar{\cdot} : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ defined by

$$\bar{1} = 1, \quad \bar{t} = 3(t - 1) \text{ for } t \geq 2.$$

One easily checks the following *admissibility* property: let $t_1, \dots, t_n \geq 1$ and $n \geq 2$ be natural numbers. If $n = 2$, let $t_1, t_2 \geq 2$. Then the following implication holds for each $t \in \mathbb{N}_+$:

$$\sum_{i=1}^n t_i \leq t + 1 \implies \sum_{i=1}^n \bar{t}_i \leq \bar{t}.$$

It is easy to see that there is no function $\mathbb{N}_+ \rightarrow \mathbb{N}_+$ with this property without this additional assumption if $n = 2$. Furthermore, the above filtration function is optimal in the sense that any other admissible filtration function is pointwise greater than or equal to the one described above.

In the literature on tame homotopy theory (see e.g. [14, 26]), homotopy, homology and cohomology groups are indexed by degrees $r + k$, where $(r - 1)$ denotes the connectivity of the spaces under consideration, and filtration functions depend on the parameter $t = k$. In our case we have $r = 1$, and in order to simplify the notation, we use the definition above which expresses the filtration function in terms of the actual degree $t = r + k$.

Definition 4.1. Let $f : A^* \rightarrow B^*$ be a map of filtered CGDAs. The map f is called a *primary weak equivalence*, if for all primes $l > q$ the map $f \otimes \text{id} : A^{*,q} \otimes \mathbb{F}_l \rightarrow B^{*,q} \otimes \mathbb{F}_l$ is a t -equivalence for all $t \geq 1$ and $q \geq \bar{t}$, i.e. the induced map in cohomology is an isomorphism in degrees less than or equal to t and injective in degree $t + 1$.

Given a natural number $k \geq 1$, we say that a primary weak equivalence $f : A^* \rightarrow B^*$ satisfies *condition k^+* , if for all primes $l > q$ the map $f \otimes \text{id} : A^{*,q} \otimes \mathbb{F}_l \rightarrow B^{*,q} \otimes \mathbb{F}_l$ is a $(t + 1)$ -equivalence for all $t \geq k$ and $q \geq \max\{\bar{t}, k + 1\}$.

Note the close relationship between filtration level and cohomological degree. The “extra degree” in condition k^+ is essential for the inductive step in the construction of a small filtered cochain model along a Postnikov decomposition of the underlying simplicial set, see the proof of Lemma 5.5.

Definition 4.2. Let (B, d_B) be a filtered CGDA, (V, d_V) a filtered cochain complex and $\tau : V^* \rightarrow B^{*+1}$ a filtration preserving cochain map of degree 1, i.e. τ satisfies the equation $d_B \circ \tau = -\tau \circ d_V$. The *free extension of B by V with twisting τ* , denoted $B \otimes_\tau \Lambda V$, is the filtered graded algebra $B \otimes \Lambda V$ equipped with the unique differential d which satisfies $d(b \otimes 1) = d_B(b) \otimes 1$ for all $b \in B$ and $d(1 \otimes v) = \tau(v) \otimes 1 + 1 \otimes d_V(v)$ for all $v \in V$ and which acts as a graded derivation.

Of particular importance are free extensions by so called elementary complexes.

Definition 4.3. Let (p, q) be a pair of natural numbers with $p, q \geq 1$. An *elementary complex of type (p, q)* is a filtered cochain complex (V^*, η) such that

$$V^{*,q'} = \begin{cases} V^{*,q} \otimes \mathbb{Q}_{q'} & \text{for } q' \geq q \\ 0 & \text{for } 0 \leq q' < q, \end{cases}$$

and $V^{*,q}$ is of the form

$$\dots \rightarrow 0 \rightarrow V^{p,q} \xrightarrow{\eta} V^{p+1,q} \rightarrow 0 \rightarrow \dots$$

where $V^{p,q}$ and $V^{p+1,q}$ are finitely generated free \mathbb{Q}_q -modules and the cokernel of η is a torsion module.

If V is an elementary complex of type (p, q) , then we let

$$\eta' : (V^{p+1,q})' := \text{Hom}(V^{p+1,q}, \mathbb{Q}_q) \rightarrow (V^{p,q})'$$

denote the dual of η . By assumption the map η' is injective. We will show that simplicial realizations of elementary complexes are Eilenberg-MacLane complexes for the group $\text{coker } \eta'$. To this end let us first transfer some elementary constructions involving cochain complexes to the filtered setting.

If $f : (V, d_V) \rightarrow (W, d_W)$ is a map of filtered cochain complexes, we denote by (C_f, d_{C_f}) the *mapping cone* of f , i.e. $C_f^{p,q} := V^{p,q} \oplus W^{p-1,q}$, where we set $W^{-1,q} = 0$, and $d_{C_f}(v, w) := (d_V(v), f(v) - d_W(w))$. Special cases are the *suspension* ΣW of W , defined as the mapping cone of $0 \rightarrow W$, and the *cone* over V , defined as the mapping cone of $\text{id} : V \rightarrow V$ and denoted by $\text{cone } V$. Note that we have a canonical short exact sequence

$$0 \rightarrow \Sigma W \rightarrow C_f \rightarrow V \rightarrow 0$$

of filtered cochain complexes.

For the notions appearing in the following proposition and its proof see [22].

Proposition 4.4. *Let (V, η) be an elementary complex of type (p, q) . Then $\|\Lambda V\|$ is a (not necessarily minimal) Eilenberg-MacLane complex of type $(\text{coker } \eta', p)$. In particular, it satisfies the Kan condition.*

Proof. By definition the set of n -simplices of $\|\Lambda V\|$ is given by

$$\|\Lambda V\|_n = \text{Mor}_{\mathcal{A}}(\Lambda V, T_n) = \text{Mor}_{\mathcal{C}}(V, T_n),$$

where \mathcal{C} denotes the category of filtered cochain complexes. We conclude that $\|\Lambda V\|$ is a simplicial abelian group, thus fulfilling the Kan condition, that fits into a pullback square of simplicial abelian groups

$$\begin{array}{ccc} \|\Lambda V\| & \xrightarrow{\quad} & (V^{p,q})' \otimes T^{p,q} \\ \downarrow & & \downarrow \text{id} \otimes d \\ (V^{p+1,q})' \otimes ZT^{p+1,q} & \xrightarrow{\eta' \otimes \text{id}} & (V^{p,q})' \otimes ZT^{p+1,q} \end{array}$$

By Corollary 3.4, $(V^{p,q})' \otimes ZT^{p+1,q}$ is an Eilenberg-MacLane complex of type $((V^{p,q})', p+1)$ and $(V^{p,q})' \otimes T^{p,q}$ is contractible. Furthermore, the right hand vertical map is surjective with kernel $(V^{p,q})' \otimes ZT^{p,q}$. It is hence a principal Kan fibration with contractible total space and fibre an Eilenberg-MacLane complex of type $((V^{p,q})', p)$, cf. [22, Lemma 18.2]. The long exact homotopy sequence for the induced fibration $\|\Lambda V\| \rightarrow (V^{p+1,q})' \otimes ZT^{p+1,q}$ shows that $\|\Lambda V\|$ is an Eilenberg-MacLane complex of type $(\text{coker } \eta', p)$. \square

Given an elementary complex (V, η) of type (p, q) we will show that the simplicial realization functor transforms the sequence

$$\Sigma V \rightarrow \text{cone } V \rightarrow V$$

into a fibre sequence

$$\|\Lambda V\| \hookrightarrow \|\Lambda(\text{cone } V)\| \rightarrow \|\Lambda(\Sigma V)\|.$$

For this we note that the map of simplicial abelian groups $\|\Lambda(\text{cone } V)\| \rightarrow \|\Lambda(\Sigma V)\|$ is surjective and hence a principal Kan fibration, whose kernel can be identified with the simplicial abelian group $\|\Lambda V\|$. The last point can be checked using an explicit description of $\|\Lambda(\text{cone } V)\|$ and $\|\Lambda(\Sigma V)\|$ similar to the one of $\|\Lambda V\|$ in the proof of Proposition 4.4. This explicit description also shows that the homotopy groups of $\|\Lambda(\text{cone } V)\|$ are zero in all degrees and hence that $\|\Lambda(\text{cone } V)\|$ is contractible. By Proposition 4.4 the simplicial set $\|\Lambda(\Sigma V)\|$ is an Eilenberg-MacLane complex of type $(\text{coker } \eta', p+1)$. We conclude that the fibration

$$\|\Lambda V\| \hookrightarrow \|\Lambda(\text{cone } V)\| \rightarrow \|\Lambda(\Sigma V)\|$$

is fibre homotopy equivalent to the path-loop fibration over an Eilenberg-MacLane complex of type $(\text{coker } \eta', p+1)$. Fibrations of this kind can thus be used as building block in the Postnikov decompositions of simplicial sets.

Proposition 4.5. *Let V be an elementary complex of type (p, q) with $V^{p,q} = \mathbb{Q}_q$ and $V^{p+1,q} = 0$. Then the unit $\Psi_{\Lambda V} : \Lambda V \rightarrow T(\|\Lambda V\|)$ induces an isomorphism $H^p((\Lambda V)^{*,q}) \rightarrow H^p(T^{*,q}(\|\Lambda V\|))$ of groups that can be canonically identified with \mathbb{Q}_q .*

Proof. The generator of $H^p(T^{*,q}(ZT^{p,q})) = \mathbb{Q}_q$ is represented by the inclusion map $ZT^{p,q} \hookrightarrow T^{p,q}$ in $\text{Mor}_{\mathcal{S}}(ZT^{p,q}, T^{p,q})$. Now recall from the proof of Proposition 4.4 that $\|\Lambda V\| = (V^{p,q})' \otimes ZT^{p,q}$. We thus obtain the unit

$$\Psi_{\Lambda V} : (\Lambda V)^{p,q} \rightarrow \text{Mor}_{\mathcal{S}}((V^{p,q})' \otimes ZT^{p,q}, T^{p,q})$$

and this sends $c \in (\Lambda V)^{p,q} = \mathbb{Q}_q$ to the morphism of simplicial sets $(V^{p,q})' \otimes ZT^{p,q} \rightarrow T^{p,q}$, $\phi \otimes z_n \mapsto \phi(c) \cdot z_n$. Together with the previous remark this implies the assertion. \square

Now let X be a simplicial set, let V be an elementary complex and let

$$f : X \rightarrow \|\Lambda(\Sigma V)\|$$

be a simplicial map. We obtain a pull back square

$$\begin{array}{ccc} E_f & \longrightarrow & \|\Lambda(\text{cone } V)\| \\ \downarrow p_f & & \downarrow \\ X & \xrightarrow{f} & \|\Lambda(\Sigma V)\| \end{array}$$

The adjoint of f is a map of filtered CGDAs $\Lambda(\Sigma V) \rightarrow T(X)$ and this is induced by a filtered cochain map $f^\sharp : V \rightarrow T(X)$ of degree 1. We hence get a free extension $T(X) \otimes_{f^\sharp} \Lambda V$ and the maps $T(p_f) : T(X) \rightarrow T(E_f)$ and the composition $V \hookrightarrow \text{cone } V \rightarrow T(E_f)$ induced by the adjoint of $E_f \rightarrow \|\Lambda(\text{cone } V)\|$ induce a filtered CGDA map

$$\Gamma_f : T(X) \otimes_{f^\sharp} \Lambda V \rightarrow T(E_f).$$

The following result will be important in the next section. It is a combination of [28, Theorem II.6.4.(i)] and [28, Lemma II.8.0]. A detailed proof will be given in Section 6.

Theorem 4.6 (Tame Hirsch lemma). *Let V be an elementary complex of type (d, \bar{k}) , where $d \geq k \geq 1$, let X be a simplicial set and let $f : X \rightarrow \|\Lambda(\Sigma V)\|$ be a simplicial map. Then the induced map*

$$\Gamma_f : T(X) \otimes_{f^\sharp} \Lambda V \rightarrow T(E_f)$$

is a primary weak equivalence satisfying condition k^+ .

We conclude this section with an example which illustrates a fundamental problem with elements of degree 1 and justifies the specific form of Definition 4.1. Let V be an elementary complex of type $(1, 1)$, concentrated in degree 1 with $V^{1,1} = \mathbb{Z}$. Then $\|\Lambda V\|$ is an Eilenberg-Mac Lane complex of type $(\mathbb{Z}, 1)$ and for each $q \geq 1$, the unit $(\Lambda V)^{*,q} \rightarrow T^{*,q}(\|\Lambda V\|)$ induces isomorphisms in cohomology in all degrees. For a constant map $f : \|\Lambda V\| \rightarrow \|\Lambda(\Sigma V)\|$, we study the map

$$\Gamma_f : T(\|\Lambda V\|) \otimes \Lambda V \rightarrow T(\|\Lambda V\| \times \|\Lambda V\|)$$

that appears in the tame Hirsch lemma. This map is not a primary 2-equivalence in filtration level 1, because $H^2((T(\|\Lambda V\|) \otimes \Lambda V)^{*,1}) = 0$, but $H^2(T^{*,1}(\|\Lambda V\| \times \|\Lambda V\|)) = \mathbb{Z}$ by the Cenkl-Porter theorem. This example shows that the extra degree in property k^+ does not occur below filtration level $\bar{k} + 1$ in general.

5. SMALL COCHAIN MODELS FOR l -TAME $(\mathbb{Z}/l)^r$ -SPACES

Let l be a prime, $G := (\mathbb{Z}/l)^r$ and X a G -space. Under certain assumptions on X we shall construct a small commutative graded differential algebra over \mathbb{F}_l which is free as a graded algebra and whose cohomology is equal to $H^*(X_G; \mathbb{F}_l)$ up to a certain degree. In particular we will prove Proposition 2.1 from Section 2.

From now on it will be convenient to work exclusively in an l -local setting. This means that all filtered cochain complexes and filtered cochain algebras appearing in Sections 3 and 4 will be tensored with $\mathbb{Z}_{(l)}$ in each filtration level. Since tensoring with $\mathbb{Z}_{(l)}$ is exact, we still obtain filtered objects. For example, if V is a filtered cochain complex, then it is understood from now on that $V^{*,q}$ is a complex of $\mathbb{Z}_{(l)}$ -modules for $0 \leq q < l$ and a complex of \mathbb{Q} -modules for $q \geq l$. The Cenkl-Porter theorem now states that we get natural multiplicative isomorphisms

$$H^*(T^{*,q}(X)) \cong \begin{cases} H^*(X; \mathbb{Z}_{(l)}) & \text{for } 1 \leq q < l \\ H^*(X; \mathbb{Q}) & \text{for } q \geq l, \end{cases}$$

where filtration levels are additive under multiplication of elements. Of course, in the definition and discussion of primary weak equivalences satisfying condition k^+ , one now restricts to the single prime l and to filtration levels $q < l$. This is underlined by the notion *l -primary weak equivalence satisfying condition k^+* , which will be used in the sequel.

Definition 5.1. We call X an l -tame G -space, if X is path connected, $\pi_1(X)$ is abelian and for all $t \geq 1$ with $\bar{t} < l$ the following holds:

- 1) $\pi_t(X)$ is a finitely generated free $\mathbb{Z}_{(l)}$ -module,
- 2) $\pi_1(X)$ acts trivially on $\pi_t(X)$ and
- 3) the induced G -action on $\pi_t(X)$ is trivial.

Lemma 5.2. Let X be an l -tame G -space. Then $\pi_1(X_G)$ is abelian and for each $t \geq 2$ with $\bar{t} < l$, $\pi_t(X_G)$ is a finitely generated free $\mathbb{Z}_{(l)}$ -module on which $\pi_1(X_G)$ acts trivially.

Proof. We only show that $\pi_1(X_G)$ is abelian. The proofs of the other assertions are elementary and left to the reader.

The short exact sequence $1 \rightarrow \pi_1(X) \rightarrow \pi_1(X_G) \rightarrow G \rightarrow 1$ can be used to construct a map

$$\phi : G \times G \rightarrow \pi_1(X), \quad (g_1, g_2) \mapsto \bar{g}_1 \cdot \bar{g}_2 \cdot (\bar{g}_1)^{-1} \cdot (\bar{g}_2)^{-1}.$$

Here \bar{g}_1 and \bar{g}_2 are lifts of $g_1, g_2 \in G = (\mathbb{Z}/l)^r$ to $\pi_1(X_G)$. The map ϕ is well-defined because the conjugation action of G on $\pi_1(X)$ is trivial. Furthermore, ϕ is \mathbb{Z}/l -bilinear. Because G is a torsion group and $\pi_1(X)$ is a free $\mathbb{Z}_{(l)}$ -module, ϕ must be the zero map and hence $\pi_1(X_G)$ is abelian. \square

Later we will need the following result, which is important to study maps between cochain complexes over $\mathbb{Z}_{(l)}$.

Lemma 5.3. *Let l be a prime and let V^* and W^* be (unfiltered) cochain complexes over $\mathbb{Z}_{(l)}$ that are torsion free $\mathbb{Z}_{(l)}$ -modules in each degree. Furthermore assume that $H^*(V^*)$ and $H^*(W^*)$ are finitely generated $\mathbb{Z}_{(l)}$ -modules in each degree.*

Now let $f : V^ \rightarrow W^*$ be a map of cochain complexes so that $f \otimes \text{id} : V^* \otimes \mathbb{F}_l \rightarrow W^* \otimes \mathbb{F}_l$ is a t -equivalence for some $t \geq 1$. Then $f^* : V^* \rightarrow W^*$ is itself a t -equivalence.*

Proof. The short exact sequence

$$0 \rightarrow \Sigma W^* \rightarrow C_f \rightarrow V^* \rightarrow 0$$

induces long exact sequences in cohomology

$$\dots \rightarrow H^i(C_f \otimes M) \rightarrow H^i(V \otimes M) \xrightarrow{f^*} H^i(W \otimes M) \rightarrow \dots$$

for any $\mathbb{Z}_{(l)}$ -module M . Our assumptions imply that $H^i(C_f \otimes \mathbb{F}_l) = 0$ for all $0 \leq i \leq t + 1$. Also, C_f consists of flat $\mathbb{Z}_{(l)}$ -modules in each degree. Hence we can apply the universal coefficient theorem [29, Theorem 5.3.14] to conclude that $H^i(C_f) \otimes \mathbb{F}_l = 0$ and hence that $H^i(C_f) = 0$ for $0 \leq i \leq t + 1$. Here we use that $H^*(C_f)$ is finitely generated over $\mathbb{Z}_{(l)}$ in each degree. From this the assertion follows. \square

From now on we work in a simplicial setting and assume that we are given a Kan complex X equipped with a G -action. This can be achieved by passing to the simplicial set of singular simplices of a given G -space. As our model for the universal fibration $EG \rightarrow BG$ we take the universal covering $EG \rightarrow \|\Delta V_0\|$, where V_0 is the elementary complex of type $(1, 1)$ given on filtration level 1 by the cochain complex

$$(5.4) \quad 0 \rightarrow \mathbb{Z}_{(l)}^r \xrightarrow{\alpha \mapsto l \cdot c} \mathbb{Z}_{(l)}^r \rightarrow 0 \rightarrow 0 \rightarrow \dots,$$

compare Proposition 4.4. In this way we realize the Borel fibration $X \hookrightarrow X_G \rightarrow BG$ as a Kan fibration.

Now let X be an l -tame G -complex. We will study the Postnikov decomposition of X_G relative to BG . This leads to a commutative diagram

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ X_2 & \longrightarrow & (X_G)_2 & \longrightarrow & BG \\ \downarrow p_2 & & \downarrow P_2 & & \parallel \\ X_1 & \longrightarrow & (X_G)_1 & \xrightarrow{P_1} & BG \\ \downarrow p_1 & & \downarrow P_1 & & \parallel \\ X_0 = \star & \longrightarrow & (X_G)_0 = BG & = & BG \end{array}$$

where for all $k \geq 1$ the complexes X_k and $(X_G)_k$ are k -th stages of the Postnikov decompositions of X and X_G , each row is a fibre sequence with fibre X_k and the vertical maps p_k and P_k are Kan fibrations whose fibres are Eilenberg-MacLane complexes of type $(\pi_k(X), k)$. By the assumption that X is l -tame together with Lemma 5.2, the maps $X_k \rightarrow X_{k-1}$ and $(X_G)_k \rightarrow (X_G)_{k-1}$ are simple fibrations, if $k \geq 1$ and $\bar{k} < l$, i.e the fundamental group of the base acts trivially on the k -th homotopy group of the fibre. Therefore, by Proposition 4.4 and the subsequent remarks, we can assume that these maps fit into diagrams

$$\begin{array}{ccc}
& \|\Lambda(\text{cone } V_k)\| & \xlongequal{\quad} \|\Lambda(\text{cone } V_k)\| \\
X_k & \xrightarrow{\quad} & (X_G)_k \\
\downarrow p_k & & \downarrow P_k \\
& \|\Lambda(\Sigma V_k)\| & \xlongequal{\quad} \|\Lambda(\Sigma V_k)\| \\
X_{k-1} & \xrightarrow{f_{k-1}} & (X_G)_{k-1} \\
& \nearrow f_{k-1} & \nearrow F_{k-1}
\end{array}$$

where (V_k, η) is an elementary complex of type (k, \bar{k}) which is concentrated in degree k with $V^{k, \bar{k}} = \pi_k(X)$.

For $k = 0$ and for $k \geq 1, \bar{k} < l$, we will define filtered CGDAs M_k and E_k together with filtered CGDA maps

$$\begin{aligned}
\phi_k : E_k &\rightarrow M_k \\
\psi_k : M_k &\rightarrow T(X_k) \\
\Psi_k : E_k &\rightarrow T((X_G)_k)
\end{aligned}$$

fitting into commutative diagrams

$$\begin{array}{ccccc}
& & T(X_k) & \xleftarrow{\quad} & T((X_G)_k) \\
& \nearrow \psi_k & \uparrow & & \nearrow \Psi_k \\
M_k & \xleftarrow{\quad} & E_k & & \\
\downarrow T(p_k) & & \downarrow T(P_k) & & \\
& T(X_{k-1}) & \xleftarrow{\quad} & T((X_G)_{k-1}) & \\
& \nearrow \psi_{k-1} & \uparrow & & \nearrow \Psi_{k-1} \\
M_{k-1} & \xleftarrow{\quad} & E_{k-1} & & \\
& \nearrow \phi_{k-1} & & &
\end{array}$$

Furthermore, for $k \geq 1$ the maps $\psi_k : M_k \rightarrow T(X_k)$ and $\Psi_k : E_k \rightarrow T((X_G)_k)$ will be l -primary weak equivalences satisfying condition k^+ , the maps ψ_0 and Ψ_0 will be l -primary weak equivalences satisfying condition 1^+ and the vertical maps in the front square will be inclusions which can be used to express M_k and E_k as free extensions of M_{k-1} and E_{k-1} making the following

diagram commutative:

$$\begin{array}{ccc}
M_k & \xleftarrow{\phi_k} & E_k \\
\parallel & & \parallel \\
M_{k-1} \otimes_{\phi_{k-1} \circ \tau_k} \Lambda V_k & \xleftarrow{\phi_{k-1} \otimes \text{id}} & E_{k-1} \otimes_{\tau_k} \Lambda V_k
\end{array}$$

We define these objects by induction on k . Let $M_0^{*,q} = \mathbb{Z}_{(l)}$, concentrated in degree 0, for $0 \leq q < l$, and let $M_0^{*,q} = \mathbb{Q}$, concentrated in degree 0, for $q \geq l$. Furthermore, define $E_0 := \Lambda V_0$, where the cochain complex V_0 was defined in (5.4). The maps

$$\begin{aligned}
\phi_0 : E_0 &\rightarrow M_0 \\
\psi_0 : M_0 &\rightarrow T(X_0) = T(\star) \\
\Psi_0 : E_0 &\rightarrow T((X_G)_0) = T(\|V_0\|)
\end{aligned}$$

are the obvious ones: ϕ_0 is an isomorphism in degree 0 and Ψ_0 is the unit. The map Ψ_0 is an l -primary weak equivalence satisfying condition 1^+ by Proposition 6.6. Now let $k \geq 1$ and $\bar{k} < l$ and assume that the above objects have been constructed for $k-1$.

Lemma 5.5. *For $k \geq 1$ and $\bar{k} < l$ the map Ψ_{k-1} induces an isomorphism*

$$\Psi_{k-1}^* : H^{k+1}(E_{k-1}^{*,\bar{k}}) \cong H^{k+1}(T^{*,\bar{k}}((X_G)_{k-1})).$$

Proof. At first we notice that the cochain complexes $E_{k-1}^{*,\bar{k}}$ and $T^{*,\bar{k}}((X_G)_{k-1})$ consist of torsion free $\mathbb{Z}_{(l)}$ -modules in each degree. Moreover, $H^*(E_{k-1}^{*,\bar{k}})$ and $H^*(T^{*,\bar{k}}((X_G)_{k-1}))$ are finitely generated over $\mathbb{Z}_{(l)}$ in each degree. This holds by the Cenkli-Porter theorem and because $\pi_*((X_G)_{k-1})$ is finitely generated over $\mathbb{Z}_{(l)}$ in each degree. Remember that X is l -tame and that $\bar{k} < l$.

Now, for $k \geq 2$ the assertion follows by Lemma 5.3, because Ψ_{k-1} is an l -primary weak equivalence satisfying condition $(k-1)^+$ and hence an l -primary $(k+1)$ -equivalence in filtration level \bar{k} .

For $k = 1$ we get $E_{k-1} = \Lambda V_0$ and the composition

$$(\Lambda V_0)^{*,\bar{1}} \otimes \mathbb{F}_l \rightarrow T^{*,\bar{1}}(\|\Lambda V_0\|) \otimes \mathbb{F}_l$$

is a 1-equivalence by Proposition 6.6. An application of Lemma 5.3 shows that the induced map

$$H^2((\Lambda V_0)^{*,\bar{1}}) \rightarrow H^2(T^{*,\bar{1}}(\|\Lambda V_0\|))$$

is injective. It is hence an isomorphism as both source and target are isomorphic to $(\mathbb{Z}/l)^r$. \square

Notice that in view of the remarks at the end of Section 4 it is somewhat unexpected that we still get an isomorphism for $k = 1$.

Because $H^*(\Sigma V_k)$ is a free $\mathbb{Z}_{(l)}$ -module concentrated in degree $k+1$, Lemma 5.5 shows that the map

$$H^*((\Sigma V_k)^{*,\bar{k}}) \rightarrow H^*(\Lambda(\Sigma V_k)^{*,\bar{k}}) \rightarrow H^*(T^{*,\bar{k}}((X_G)_{k-1}))$$

induced by the adjoint of F_{k-1} can be written as the composition of Ψ_{k-1}^* and a map

$$H^*((\Sigma V_k)^{*,\bar{k}}) \rightarrow H^*(E_{k-1}^{*,\bar{k}}).$$

This map is induced by a map of filtered cochain complexes

$$\tau_k : \Sigma V_k^* \rightarrow E_{k-1}^*$$

and defines the free extensions $E_k = E_{k-1} \otimes_{\tau_k} \Lambda V_k$ and $M_k = M_{k-1} \otimes_{\phi_{k-1} \circ \tau_k} \Lambda V_k$ as well as the map $\phi_k := (\phi_{k-1} \otimes \text{id}) : E_k \rightarrow M_k$.

It remains to construct the l -primary weak equivalences satisfying condition k^+

$$\begin{aligned} \psi_k : M_k &\rightarrow T(X_k) \\ \Psi_k : E_k &\rightarrow T((X_G)_k). \end{aligned}$$

Let

$$F_{k-1}^\sharp : \Sigma V_k \rightarrow T((X_G)_{k-1})$$

be the restriction of the adjoint of F_{k-1} .

The composition

$$\Psi_{k-1} \circ \tau_k : \Sigma V_k^{*,\bar{k}} \rightarrow T^{*,\bar{k}}((X_G)_{k-1})$$

induces the same map in cohomology as the map F_{k-1}^\sharp . We denote by $\overline{F_{k-1}}$ the adjoint of $\Psi_{k-1} \circ \Lambda(\tau_k) : \Lambda(\Sigma V_k) \rightarrow E_{k-1} \rightarrow T((X_G)_{k-1})$ and conclude that the maps F_{k-1} and $\overline{F_{k-1}}$ are homotopic as maps of simplicial sets $(X_G)_{k-1} \rightarrow \|\Lambda(\Sigma V_k)\|$. This follows, because $\|\Lambda(\Sigma V_k)\|$ is an Eilenberg-MacLane complex of type $(\pi_k(X), k+1)$ and because after applying the functor $T^{*,\bar{k}}$ and precomposing with the unit

$$\Psi_{\Lambda(\Sigma V_k)}|_{\Sigma V_k^{*,\bar{k}}} : \Sigma V_k^{*,\bar{k}} \rightarrow T^{*,\bar{k}}(\|\Lambda(\Sigma V_k)\|),$$

we precisely obtain the maps F_{k-1}^\sharp and $\Psi_{k-1} \circ \tau_k$ considered at the beginning of this paragraph, see also Proposition 4.5.

By precomposing with the inclusion $X_{k-1} \rightarrow (X_G)_{k-1}$ we also obtain a homotopy between f_{k-1} and $\overline{f_{k-1}}$ where the last map is adjoint to $\psi_{k-1} \circ \phi_{k-1} \circ \Lambda(\tau_k)$. It follows that there are homotopy equivalences

$$\begin{aligned} \xi : X_k &\simeq E_{\overline{f_{k-1}}} \\ \Xi : (X_G)_k &\simeq E_{\overline{F_{k-1}}} \end{aligned}$$

making the diagram

$$\begin{array}{ccccc} X_k & \xrightarrow{\quad} & (X_G)_k & & \\ & \searrow p_k & & \swarrow P_k & \\ & & X_{k-1} \longrightarrow (X_G)_{k-1} & & \\ & \nearrow & & \nwarrow & \\ E_{\overline{f_{k-1}}} & \xrightarrow{\quad} & E_{\overline{F_{k-1}}} & & \end{array}$$

$\xi \downarrow \quad \quad \quad \Xi \downarrow$

commutative. We now define Ψ_k and ψ_k as the compositions

$$\begin{aligned} \Psi_k : E_{k-1} \otimes_{\tau_k} \Lambda V_k &\xrightarrow{\Psi_{k-1} \otimes \text{id}} T((X_G)_{k-1}) \otimes_{\Psi_{k-1} \circ \tau_k} \Lambda V_k \xrightarrow{\Gamma_{\overline{F_{k-1}}}} T(E_{\overline{F_{k-1}}}) \xrightarrow{T(\Xi)} T((X_G)_k) \\ \psi_k : M_{k-1} \otimes_{\phi_{k-1} \circ \tau_k} \Lambda V_k &\xrightarrow{\psi_{k-1} \otimes \text{id}} T(X_{k-1}) \otimes_{\psi_{k-1} \circ \phi_{k-1} \circ \tau_k} \Lambda V_k \xrightarrow{\Gamma_{\overline{f_{k-1}}}} T(E_{\overline{f_{k-1}}}) \xrightarrow{T(\xi)} T(X_k) \end{aligned}$$

By the tame Hirsch lemma and Lemma 6.3, these maps are l -primary weak equivalences satisfying condition k^+ , since a primary weak equivalence satisfying condition $(k-1)^+$ is also a primary weak equivalence satisfying condition k^+ and $(X_G)_{k-1}$ is connected by assumption. This finishes the construction of Ψ_k and ψ_k .

We now formulate the main result of this section.

Theorem 5.6. *Let X be an l -tame G -space and let $k \geq 1$ be a number with $\bar{k} < l$. Then there are (unfiltered) commutative graded differential algebras (E^*, d_E) and (M^*, d_M) over \mathbb{F}_l with the following properties:*

- 1) *as graded algebras $E^* = \mathbb{F}_l[t_1, \dots, t_r] \otimes \Lambda(s_1, \dots, s_r) \otimes M^*$ where each t_i is of degree 2 and each s_i is of degree 1.*
- 2) *the differential d_E is zero on $\mathbb{F}_l[t_1, \dots, t_r] \otimes \Lambda(s_1, \dots, s_r) \otimes 1$ and the map $E^* \rightarrow M^*, t_i, s_i \mapsto 0$, is a cochain map.*
- 3) *M^* is free as a graded algebra with generators in degrees $1, \dots, k$. As generators in degree $i \in \{1, \dots, k\}$ we can take the elements of a basis of the free $\mathbb{Z}_{(l)}$ -module $\pi_i(X)$.*
- 4) *the cohomology algebra of E^* is multiplicatively isomorphic to $H^*(X_G; \mathbb{F}_l)$ up to degree k .*
- 5) *assume that $H^*(X_G; \mathbb{F}_l)$ vanishes in degrees k and $k+1$. Then each monomial in t_1, \dots, t_r of cohomological degree at least k represents the zero class in $H^*(E)$.*

Proof. After k steps the above process leads to a commutative diagram

$$\begin{array}{ccc} E_k & \xrightarrow{\phi_k} & M_k \\ \downarrow \Psi_k & & \downarrow \psi_k \\ T((X_G)_k) & \longrightarrow & T(X_k) \end{array}$$

where the vertical arrows are l -primary weak equivalences satisfying condition k^+ . We can restrict this diagram to filtration level \bar{k} to obtain a commutative diagram of \mathbb{F}_l -cochain complexes

$$\begin{array}{ccc} E_k^{*, \bar{k}} \otimes \mathbb{F}_l & \longrightarrow & M_k^{*, \bar{k}} \otimes \mathbb{F}_l \\ \downarrow & & \downarrow \\ T^{*, \bar{k}}((X_G)_k) \otimes \mathbb{F}_l & \longrightarrow & T^{*, \bar{k}}(X_k) \otimes \mathbb{F}_l. \end{array}$$

The vertical arrows in this diagram are k -equivalences. By its inductive construction the cochain complex $E_k^{*, \bar{k}}$ can be written as

$$E_k^{*, \bar{k}} = (((\Lambda V_0 \otimes_{\tau_1} \Lambda V_1) \otimes \dots) \otimes_{\tau_k} \Lambda V_k)^{*, \bar{k}}$$

where V_0 is an elementary complex of type $(1, 1)$ and V_t is an elementary complex of type (t, \bar{t}) which is concentrated in degree t for $t \in \{1, \dots, k\}$.

We set

$$W_0 := \Lambda(V_0^{*, 1} \otimes \mathbb{F}_l), \quad W_t := \Lambda(V_t^{*, \bar{t}} \otimes \mathbb{F}_l) \text{ for } 1 \leq t \leq k,$$

regarded as unfiltered \mathbb{F}_l -cochain algebras. We hence get an unfiltered \mathbb{F}_l -cochain algebra

$$E^* = ((W_0 \otimes_{\tau_1} W_1) \otimes \dots) \otimes_{\tau_k} W_k.$$

Note that $W_0 = \mathbb{F}_l[t_1, \dots, t_r] \otimes \Lambda(s_1, \dots, s_r)$ with trivial differential. We denote the evaluation $t_i, s_i \mapsto 0$ by ϕ and obtain an \mathbb{F}_l -cochain algebra

$$M^* = (W_1 \otimes_{\phi \circ \tau_2} W_2 \otimes \dots) \otimes_{\phi \circ \tau_k} W_k.$$

It follows that M^* is of the form described in Theorem 5.6, that

$$E^* = \mathbb{F}_l[t_1, \dots, t_r] \otimes \Lambda(s_1, \dots, s_r) \otimes M^*$$

as graded algebras and that $E^* \rightarrow M^*, t_i, s_i \mapsto 0$, is a cochain map.

It remains to study the relation between $H^*(E)$ and $H^*(X_G; \mathbb{F}_l)$ as stated in parts 4) and 5) of Theorem 5.6. Here we have to keep in mind that the product

$$(E_k^{p_1, q_1} \otimes \mathbb{F}_l) \otimes (E_k^{p_2, q_2} \otimes \mathbb{F}_l) \rightarrow E_k^{p_1+p_2, q_1+q_2} \otimes \mathbb{F}_l$$

is the zero map, if $q_1 + q_2 \geq l$. Therefore we have to be somewhat careful when comparing the multiplications in the unfiltered algebra E^* with those in the filtered algebra E_k^* .

Let $B_0 \subset V_0^{*,1}$ and $B_t \subset V_t^{*,\bar{t}}$ be homogenous $\mathbb{Z}_{(l)}$ -module bases for $1 \leq t \leq k$. We consider the set S of monomials of homological degree at most k in elements of $B_0 \cup \dots \cup B_k$. The admissibility of the filtration function implies that $S \subset E_k^{*,\bar{k}}$. Let $Z^* \subset E_k^{*,\bar{k}}$ be the $\mathbb{Z}_{(l)}$ -subcomplex which is generated by the monomials in S and their coboundaries.

We obtain inclusions $Z^* \otimes \mathbb{F}_l \rightarrow E^*$ and $Z^* \otimes \mathbb{F}_l \rightarrow E_k^{*,\bar{k}} \otimes \mathbb{F}_l$ of cochain complexes both of which are k -equivalences. This leads to an additive isomorphism

$$\alpha : H^{\leq k}(E^*) \cong H^{\leq k}(E_k^{*,\bar{k}} \otimes \mathbb{F}_l).$$

Consider the composition

$$\beta : H^*(E_k^{*,\bar{k}} \otimes \mathbb{F}_l) \rightarrow H^*(T^{*,\bar{k}}((X_G)_k) \otimes \mathbb{F}_l) \xrightarrow{\cong} H^*((X_G)_k; \mathbb{F}_l) \rightarrow H^*(X_G; \mathbb{F}_l)$$

where the second map is the Cenkl-Porter map and the last map is induced by the canonical map $X_G \rightarrow (X_G)_k$ to the k -th Postnikov piece. As a composition of k -equivalences the map β is itself a k -equivalence.

In summary, we obtain an additive isomorphism

$$\beta \circ \alpha : H^{\leq k}(E^*) \cong H^{\leq k}(E_k^{*,\bar{k}} \otimes \mathbb{F}_l) \cong H^{\leq k}(X_G; \mathbb{F}_l).$$

In order to study the multiplicative behaviour of $\beta \circ \alpha$, let $c_1 \in H^{p_1}(E)$ and $c_2 \in H^{p_2}(E)$, where $1 \leq p_1, p_2 \leq k$ and $p_1 + p_2 \leq k$. Then, by the construction of $E_k^{*,\bar{k}}$, the elements $\alpha(c_1)$ and $\alpha(c_2)$ can be represented by cocycles $\gamma_1 \in E_k^{p_1, \bar{p}_1} \otimes \mathbb{F}_l$ and $\gamma_2 \in E_k^{p_2, \bar{p}_2} \otimes \mathbb{F}_l$ and hence $\alpha(c_1 \cdot c_2)$ can be represented by the cocycle $\gamma_1 \cdot \gamma_2 \in E_k^{p_1+p_2, \bar{p}_1+\bar{p}_2} \otimes \mathbb{F}_l \subset E_k^{p_1+p_2, \bar{p}_1+\bar{p}_2} \otimes \mathbb{F}_l \subset E_k^{p_1+p_2, \bar{k}} \otimes \mathbb{F}_l$. Here we have used the assumption $p_1 + p_2 \leq k$ and the properties of our filtration function. The proof of assertion 4) is hence complete, because the diagram

$$\begin{array}{ccc} H^{p_1}(E_k^{*,\bar{p}_1}) \otimes H^{p_2}(E_k^{*,\bar{p}_2}) & \xrightarrow{\text{mult.}} & H^{p_1+p_2}(E_k^{*,\bar{p}_1+\bar{p}_2}) \\ \downarrow \beta \otimes \beta & & \downarrow \beta \\ H^{p_1}(X_G; \mathbb{F}_l) \otimes H^{p_2}(X_G; \mathbb{F}_l) & \xrightarrow{\cup} & H^{p_1+p_2}(X_G; \mathbb{F}_l) \end{array}$$

commutes.

Now assume that $H^*(X_G; \mathbb{F}_l)$ vanishes in degrees k and $k+1$. Let $\mu \in E^*$ be a monomial in t_1, \dots, t_r of cohomological degree at least k . Because the elements t_1, \dots, t_r are cocycles in E^2

and because the differential acts as a derivation, it is enough to assume that μ has cohomological degree k or $k + 1$. If the cohomological degree is k , then μ represents the zero class in $H^*(E)$, because $H^k(E^*) \cong H^k(X_G; \mathbb{F}_l) = 0$. If the cohomological degree is $k + 1$, then we argue as follows. The elements $t_1, \dots, t_r \in E^2$ correspond to cocycles $\bar{t}_1, \dots, \bar{t}_r \in V_0^{2,1} \otimes \mathbb{F}_l$. By the choice of our filtration function, we obtain a monomial $\bar{\mu} \in E^{k+1, \bar{k}} \otimes \mathbb{F}_l$ corresponding to μ . Because $\beta : H^{k+1}(E_k^{*, \bar{k}} \otimes \mathbb{F}_l) \rightarrow H^{k+1}(X_G; \mathbb{F}_l) = 0$ is injective, there is an element $\nu \in E_k^{k, \bar{k}} \otimes \mathbb{F}_l$ with $\bar{\mu} = d(\nu)$, where d is the differential on $E_k^{*, \bar{k}} \otimes \mathbb{F}_l$. But $\nu \in Z^* \otimes \mathbb{F}_l$ by construction of Z^* . Because $Z^* \otimes \mathbb{F}_l \rightarrow E^*$ is a cochain map, we conclude that μ is a cocycle in E^* , which finishes the proof of claim 5). \square

If instead we start with a G -space X which is not l -tame, then Theorem 5.6 can still be applied in many cases. To illustrate this let X be a connected nilpotent Kan complex with abelian fundamental group and let G act on X . We denote by $X_{(l)}$ the l -localization of X , see [7]. By the assumptions on X we have $\pi_t(X_{(l)}) = \pi_t(X) \otimes \mathbb{Z}_{(l)}$ for $t \geq 1$. Furthermore, by the functoriality of the l -localization, $X_{(l)}$ is again a G -space and the canonical G -equivariant map $X \rightarrow X_{(l)}$ induces an isomorphism

$$H^*((X_{(l)})_G; \mathbb{F}_l) \rightarrow H^*(X_G; \mathbb{F}_l)$$

by a spectral sequence argument. Hence, if $X_{(l)}$ is l -tame, then the preceding discussion applies to $(X_{(l)})_G$ and for any $k \geq 1$ with $\bar{k} < l$ we obtain a commutative graded differential algebra E over \mathbb{F}_l which calculates the \mathbb{F}_l -cohomology of X_G up to degree k .

We specialize this discussion to the G -space $X = S^{n_1} \times \dots \times S^{n_k}$ appearing in Section 2. Let $l > 3 \dim X$. For each $t \geq 1$ with $\bar{t} < l$ this assumption implies

$$\pi_t(S^{n_j}) \otimes \mathbb{Z}_{(l)} = \begin{cases} \mathbb{Z}_{(l)} & \text{for } j > k_e \text{ and } t = n_j \\ \mathbb{Z}_{(l)} & \text{for } j \leq k_e \text{ and } t = n_j \text{ or } t = 2n_j - 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $j \in \{1, \dots, k\}$. Hence $\pi_t(X_{(l)})$ is a finitely generated free $\mathbb{Z}_{(l)}$ module of rank at most k . Because $l > 3 \dim X$, we have $l - 1 > k$. This implies that G acts trivially on $\pi_t(X_{(l)})$, because the rational representation theory [27] of the group \mathbb{Z}/l tells us that the smallest dimension of a nontrivial rational \mathbb{Z}/l -representation is $l - 1$. We conclude that $X_{(l)}$ is an l -tame G -space in the sense of Definition 5.1.

Theorem 5.6 and the subsequent discussion yield an \mathbb{F}_l -cochain algebra E^* whose cohomology is isomorphic to $H^*(X_G; \mathbb{F}_l)$ in degrees less than or equal to $\dim X + 1$ and with the described behaviour of monomials in t_1, \dots, t_r . This uses $l > 3 \dim X = \overline{\dim X} + 1$ so that we can work with $k := \dim X + 1$.

It follows from Theorem 5.6 and the partial calculation of $\pi_*((S^{n_j})_{(l)})$ displayed above that E^* has the form described in Proposition 2.1.

6. APPENDIX: PROOF OF THE TAME HIRSCH LEMMA

The material in this appendix is based on Sections II.7 and II.8 of the diploma thesis of Sørensen [28]. Because the proof of the tame Hirsch lemma, a result which is fundamental for our approach, is quite delicate, we provide the following detailed exposition.

Theorem 4.6 is first proven under the additional assumption that the unit

$$\Psi_{\Lambda V} : \Lambda V \rightarrow T(\|\Lambda V\|)$$

is a primary weak equivalence satisfying condition k^+ . We will show in Proposition 6.6 below that there is no loss of generality in assuming this.

We work in several steps. At first, we consider the untwisted case, i.e. when f is a constant map. Then $E_f = X \times \|\Lambda V\|$ and Γ_f is induced by the projections $X \times \|\Lambda V\| \rightarrow X$, $X \times \|\Lambda V\| \rightarrow \|\Lambda V\|$, the unit $\Psi_{\Lambda V}$ and the product of forms in $T(X \times \|\Lambda V\|)$.

Now consider the commutative diagram of filtered graded differential algebras

$$\begin{array}{ccccc}
 (6.1) & T(X) \otimes \Lambda V & \xrightarrow{\text{id} \otimes \Psi_{\Lambda V}} & T(X) \otimes T(\|\Lambda V\|) & \xrightarrow{\wedge} & T(X \times \|\Lambda V\|) \\
 & \downarrow & & \downarrow & & \downarrow \rho \\
 & T(X) \overline{\otimes} \Lambda V & \xrightarrow{\text{id} \overline{\otimes} \Psi_{\Lambda V}} & T(X) \overline{\otimes} T(\|\Lambda V\|) & & \\
 & & & \downarrow \rho & & \\
 & & & ((T \overline{\otimes} C)(X)) \overline{\otimes} ((T \overline{\otimes} C)(\|\Lambda V\|)) & \xrightarrow{\wedge \overline{\otimes} \times} & (T \overline{\otimes} C)(X \times \|\Lambda V\|) \\
 & & & \uparrow \sigma & & \uparrow \sigma \\
 & & & C(X) \overline{\otimes} C(\|\Lambda V\|) & \xrightarrow{\times} & C(X \times \|\Lambda V\|)
 \end{array}$$

where the maps ρ and σ are taken from the zig-zag sequence (3.6) and the bar indicates the filtrationwise tensor product.

Lemma 6.2. *The canonical map*

$$T(X) \otimes \Lambda V \rightarrow T(X) \overline{\otimes} \Lambda V$$

is a primary weak equivalence satisfying condition k^+ .

Proof. For $n \geq 0$ let $(\Lambda^{(n)}V) \subset \Lambda V$ be the graded \mathbb{Q} -submodule generated by monomials consisting of exactly n factors in V . Note that $\Lambda^{(n)}V$ inherits a filtration from ΛV and that $(\Lambda^{(n)}V)^{*,q} = 0$ for $q < n\bar{k}$ and $(\Lambda^{(n)}V)^{*,q} = (\Lambda^{(n)}V)^{*,n\bar{k}} \otimes \mathbb{Q}_q$ for $q \geq n\bar{k}$. Furthermore, the differential on ΛV restricts to a differential on $\Lambda^{(n)}V$.

For $q \geq 1$ we have direct sum decompositions

$$\begin{aligned}
 (T(X) \otimes (\Lambda V))^{*,q} &= \bigoplus_{n \geq 0, n\bar{k} \leq q} T^{*,q-n\bar{k}}(X) \otimes (\Lambda^{(n)}V)^{*,n\bar{k}} \otimes \mathbb{Q}_q \\
 (T(X) \overline{\otimes} (\Lambda V))^{*,q} &= \bigoplus_{n \geq 0, n\bar{k} \leq q} T^{*,q}(X) \otimes (\Lambda^{(n)}V)^{*,q}.
 \end{aligned}$$

It is therefore enough to investigate the canonical map

$$T^{*,q-n\bar{k}}(X) \otimes (\Lambda^{(n)}V)^{*,n\bar{k}} \otimes \mathbb{F}_l \rightarrow T^{*,q}(X) \otimes (\Lambda^{(n)}V)^{*,q} \otimes \mathbb{F}_l$$

for each $n \geq 0$ with $n\bar{k} \leq q$ and any prime $l > q$. By the algebraic Künneth formula for chain complexes over the field \mathbb{F}_l , the induced map in cohomology splits in degree $t \geq 0$ into a direct sum of maps

$$\omega : H^i(T^{*,q-n\bar{k}}(X) \otimes \mathbb{F}_l) \otimes H^j((\Lambda^{(n)}V)^{*,n\bar{k}} \otimes \mathbb{F}_l) \rightarrow H^i(T^{*,q}(X) \otimes \mathbb{F}_l) \otimes H^j((\Lambda^{(n)}V)^{*,q} \otimes \mathbb{F}_l),$$

where $i, j \geq 0$ and $i + j = t$. We have $H^j((\Lambda^{(n)}V)^{*,n\bar{k}} \otimes \mathbb{F}_l) = H^j((\Lambda^{(n)}V)^{*,q} \otimes \mathbb{F}_l)$, because $l > q \geq n\bar{k}$. Now, if $q > n\bar{k}$, then the map $H^*(T^{*,q-n\bar{k}}(X) \otimes \mathbb{F}_l) \rightarrow H^*(T^{*,q}(X) \otimes \mathbb{F}_l)$ is an isomorphism by the addendum to the Cenkl-Porter Theorem 3.5 and hence ω is an isomorphism.

We study the remaining case $q = n\bar{k}$. Because $H^*((\Lambda^{(n)}V)^{*,n\bar{k}} \otimes \mathbb{F}_l)$ and $H^*((\Lambda^{(n)}V)^{*,q} \otimes \mathbb{F}_l)$ are concentrated in degrees at least nd , the map ω is an isomorphism if $i + j \leq nd$ and injective, if $i + j \leq nd + 1$. Recall that for $i = 1$ and $q = n\bar{k}$, the source of ω is the zero module.

First let $t \geq 1$ and $q \geq \bar{t}$. This implies $n \geq 1$.

If $n = 1$, we get $t \leq nd$, because our assumption $\bar{t} \leq q = 1 \cdot \bar{k}$ implies $t \leq k$ and hence $t \leq d$ (recall that $d \geq k$ by assumption).

If $n \geq 2$, then we have $t + 1 \leq nd$. For if we assume $t \geq nd$, then the obvious estimate $nd + ((t+1) - nd) \leq t+1$ and the admissibility of the filtration function imply $n\bar{d} + (t+1) - nd \leq \bar{t}$ and hence $n\bar{d} < \bar{t}$. If $d = 1$, then this requires the assumption $n \geq 2$. But this contradicts the assumed estimate $\bar{t} \leq q = n\bar{k} \leq n\bar{d}$.

These considerations show that ω is a t -equivalence, if $t \geq 1$ and $q \geq \bar{t}$.

Now observe, that if $q \geq \bar{k} + 1$, then the assumption $q = n\bar{k}$ automatically implies $n \geq 2$. Hence, the preceding estimate for the case $n \geq 2$ shows that under this assumption, ω is actually a $(t+1)$ -equivalence. This completes the proof of Lemma 6.2. \square

To continue the proof of Theorem 4.6 for constant f we observe the following points. Firstly, the map $\text{id} \otimes \Psi_{\Lambda V} : T(X) \otimes \Lambda V \rightarrow T(X) \otimes T(\|\Lambda V\|)$ is a primary weak equivalence satisfying condition k^+ . This follows from the Künneth formula [29, Lemma 5.3.1] and the fact that by assumption $\Psi_{\Lambda V}$ is a primary weak equivalence satisfying condition k^+ . Secondly, the universal coefficient theorem [29, Theorem 5.3.14] implies that for each $q \geq 1$ and each prime $l > q$, the maps ρ and σ induce isomorphisms in cohomology after tensoring the cochain complexes with \mathbb{F}_l . Thirdly, the cross product map

$$C^*(X; \mathbb{F}_l) \otimes C^*(\|\Lambda V\|; \mathbb{F}_l) \rightarrow C^*(X \times \|\Lambda V\|; \mathbb{F}_l)$$

induces an isomorphism of cohomology groups for any prime l . This uses the Künneth formula for cohomology [29, Theorem 5.6.1] and the fact that for any prime l , the homology $H_*(\|\Lambda V\|; \mathbb{F}_l)$ is finitely generated in each degree. Recall that $\|\Lambda V\|$ is an Eilenberg-MacLane complex for a finitely generated $\mathbb{Q}_{\bar{k}}$ -module, see Proposition 4.4. This completes the proof of Theorem 4.6 for constant f .

The next step of the proof of Theorem 4.6 deals with the case when f is homotopic to a constant map. But before we give a proof of this case, we will show the second main technical result of this section besides Lemma 6.2.

Lemma 6.3. *Let A and B be filtered CGDAs and let V be an elementary complex of type (d, \bar{k}) with $d \geq k \geq 1$. Assume that $\tau : V \rightarrow A$ is a filtered cochain map of degree 1 and let $f : A \rightarrow B$ be a primary weak equivalence satisfying condition k^+ so that $f : A^{*,0} \rightarrow B^{*,0}$ is an isomorphism. Then the induced map*

$$A \otimes_{\tau} \Lambda V \rightarrow B \otimes_{f \circ \tau} \Lambda V$$

is a primary weak equivalence satisfying condition k^+ .

Proof. For $q \geq 1$ let us consider the decreasing filtrations \mathcal{F} of $(A \otimes_{\tau} \Lambda V)^{*,q}$ and $(B \otimes_{f \circ \tau} \Lambda V)^{*,q}$ defined for $\gamma \geq 0$ by

$$\begin{aligned} \mathcal{F}_{\gamma}(A \otimes_{\tau} \Lambda V)^{*,q} &= (A^{\geq \gamma} \otimes_{\tau} \Lambda V)^{*,q} \\ \mathcal{F}_{\gamma}(B \otimes_{f \circ \tau} \Lambda V)^{*,q} &= (B^{\geq \gamma} \otimes_{f \circ \tau} \Lambda V)^{*,q}. \end{aligned}$$

Let $l > q$ be a prime. The direct sum decomposition $\Lambda V = \bigoplus \Lambda^{(n)}V$ from the proof of Lemma 6.2 leads to the following direct sum decompositions of the E_1 -terms of the resulting spectral

sequences with \mathbb{F}_l -coefficients:

$$\begin{aligned} E_1^{i,j}(A) &= \bigoplus_{n \geq 0, \bar{n} \leq q} (A^{i,q-\bar{n}} \otimes \mathbb{F}_l) \otimes H^j((\Lambda^{(n)}V)^{*,\bar{n}} \otimes \mathbb{F}_l) \\ E_1^{i,j}(B) &= \bigoplus_{n \geq 0, \bar{n} \leq q} (B^{i,q-\bar{n}} \otimes \mathbb{F}_l) \otimes H^j((\Lambda^{(n)}V)^{*,\bar{n}} \otimes \mathbb{F}_l). \end{aligned}$$

On the E_2 -terms we hence get the direct sum decompositions

$$\begin{aligned} E_2^{i,j}(A) &= \bigoplus_{n \geq 0, \bar{n} \leq q} H^i(A^{*,q-\bar{n}} \otimes \mathbb{F}_l) \otimes H^j((\Lambda^{(n)}V)^{*,\bar{n}} \otimes \mathbb{F}_l) \\ E_2^{i,j}(B) &= \bigoplus_{n \geq 0, \bar{n} \leq q} H^i(B^{*,q-\bar{n}} \otimes \mathbb{F}_l) \otimes H^j((\Lambda^{(n)}V)^{*,\bar{n}} \otimes \mathbb{F}_l), \end{aligned}$$

but notice that the differentials d_2 on the E_2 -terms do not in general restrict to the single summands (with varying i and j) unless τ is zero.

We will now prove that the map of the E_2 -terms induced by f is a t -equivalence, if $t \geq 1$ and $q \geq \bar{t}$ and a $(t+1)$ -equivalence, if $t \geq k+1$ and $q \geq \bar{t}$ or if $t = k$ and $q \geq \bar{k}+1$, where $t = i+j$ is the cohomological degree. This implies the statement of Proposition 6.3 by induction over the pages in the spectral sequence.

First notice that $H^*((\Lambda^{(n)}V)^{*,\bar{n}} \otimes \mathbb{F}_l)$ is concentrated in degrees at least nd so that it suffices to show that

$$\omega : A^{*,q-\bar{n}} \otimes \mathbb{F}_l \rightarrow B^{*,q-\bar{n}} \otimes \mathbb{F}_l$$

is a $(t-nd)$ -equivalence, respectively a $((t+1)-nd)$ -equivalence in the relevant cases. Since the map ω is itself an isomorphism for $q-\bar{n} = 0$ and the map ω is a 1-equivalence for $q-\bar{n} \geq 1 = \bar{1}$, the map ω is a $((t+1)-nd)$ -equivalence, if $-1 \leq ((t+1)-nd) \leq 1$. Hence we will assume from now on that $t-nd \geq 1$. Because the case $n = 0$ is trivial, we will also assume that $n \geq 1$.

We start by analyzing the case $t \geq 1$ and $q \geq \bar{t}$.

We will show at first that ω is a $(t-nd)$ -equivalence. Because $t-nd \geq 1$ we can use the admissibility of the filtration function and the inequality $1+nd+(t-nd) \leq t+1$ to conclude $\bar{1}+n\bar{d}+\overline{t-nd} \leq \bar{t}$ which implies $q-n\bar{k} \geq \overline{t-nd}$ (remember $k \leq d$). Hence ω is a $(t-nd)$ -equivalence by the assumption on f .

If in addition $n \geq 2$ or if $n = 1$ and $d > 1$, then the inequality $nd+((t+1)-nd) \leq t+1$ and the admissibility of the filtration function imply $q-n\bar{k} \geq \overline{(t+1)-nd}$ and ω is even a $((t+1)-nd)$ -equivalence.

Now let $n = 1$ and $d = 1$. This implies $k = 1$. We will show that ω is still a $((t+1)-nd)$ -equivalence, if $t \geq k+1$, $q \geq \bar{t}$, or if $t = k$, $q \geq \bar{t}+1$.

Let us first concentrate on the case $t \geq k+1$ and $q \geq \bar{t}$. If $t \geq k+2$, then $q-n\bar{k} = q-1 \geq \bar{t}-1 \geq \overline{t-1}$ and $t-1 \geq k+1$, so that ω is a $((t-1)+1)$ -equivalence by the assumption on f . In other words, ω is a $((t+1)-nd)$ -equivalence as desired. If $t = k+1$, then (as $q \geq \bar{t}$) we get $q-n\bar{k} \geq \bar{k}+1-1 = \bar{k}+1$ (remember $k = 1$), so that ω is a $(k+1)$ -equivalence by assumption on f and hence ω is a $((t+1)-nd)$ -equivalence.

In the case $t = k$, $q \geq \bar{t}+1$, we get $q-n\bar{k} = q-1 \geq \bar{t}$ so that ω is a t -equivalence by the assumption on f and hence ω is again a $((t+1)-nd)$ -equivalence.

This finishes the proof of Lemma 6.3. □

In order to proceed with the proof of Theorem 4.6, we assume $f \simeq \text{const}$ and choose a simplicial map $H : X \times I \rightarrow \|\Lambda(\Sigma V)\|$ with $H \circ i_0 = f$ and $H \circ i_1 = \text{const}$, where I is the simplicial interval and $i_0, i_1 : X \rightarrow X \times I$ are the canonical inclusions. For $\epsilon = 0$ and $\epsilon = 1$, we obtain a commutative diagram

$$\begin{array}{ccccc} E_\epsilon & \xrightarrow{I_\epsilon} & E_H & \longrightarrow & \|\Lambda(\text{cone } V)\| \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{i_\epsilon} & X \times I & \xrightarrow{H} & \|\Lambda(\Sigma V)\| \end{array}$$

where the horizontal maps of the first square are *strong equivalences*. By definition this means that after applying the Cenkli-Porter functor T , one gets isomorphisms in cohomology in each degree and in each filtration level. Using the naturality of the construction of the map Γ , we hence get an induced commutative diagram

$$\begin{array}{ccc} T(X) \otimes \Lambda(V) & \xrightarrow{\Gamma_{\text{const}}} & T(E_{\text{const}}) \\ T(i_1) \otimes \text{id} \uparrow & & \uparrow T(I_1) \\ T(X \times I) \otimes_{H\#} \Lambda(V) & \xrightarrow{\Gamma_H} & T(E_H) \\ T(i_0) \otimes \text{id} \downarrow & & \downarrow T(I_0) \\ T(X) \otimes_{f\#} \Lambda(V) & \xrightarrow{\Gamma_f} & T(E_f) \end{array}$$

For any $q \geq 1$ and $l > q$ the vertical maps in this diagram restricted to filtration level q induce isomorphisms of cohomology groups with \mathbb{F}_l -coefficients. This is clear for the right hand column. For the left hand column one argues with the spectral sequences introduced at the beginning of the proof of Lemma 6.3.

Hence, because Γ_{const} is a primary weak equivalence satisfying condition k^+ , the same is true for the map Γ_f .

Now these special cases are used to show the tame Hirsch lemma for arbitrary simplicial sets X by a Mayer-Vietoris argument and induction over the m -skeleta of X .

We consider simplicial maps $f : X \rightarrow \|\Lambda(\Sigma V)\|$ where X is a simplicial set and V is an elementary complex of type (d, \bar{k}) with $d \geq k \geq 1$. In the following we leave V fixed, but consider varying X .

First, the tame Hirsch lemma holds, if X is a disjoint union of simplicial m -simplices, $m \geq 0$. This follows from the preceding line of arguments, because in this case f is homotopic to a constant map. Recall that $\|\Lambda(\Sigma V)\|$ is connected and fulfills the Kan condition by Proposition 4.4. In particular, the tame Hirsch lemma holds, if X is a disjoint union of 0-dimensional simplicial sets. We now assume that the tame Hirsch lemma has been proven for any $(m-1)$ -dimensional simplicial set X , where $m \geq 1$. If X is m -dimensional, we write X as a push out

$$\begin{array}{ccc} \coprod \partial\Delta[m] & \longrightarrow & \coprod \Delta[m] \\ \downarrow & & \downarrow \\ X^{m-1} & \longrightarrow & X \end{array}$$

and do the same for E_f . Here $\partial\Delta[m]$ denotes the simplicial $(m-1)$ -sphere. For $q \geq 1$ we consider the following commutative diagram, where the columns are Mayer-Vietoris sequences

and the horizontal arrows are induced by Γ_f or appropriate restrictions thereof:

$$(6.4) \quad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ (T(X) \otimes_{f\#} \Lambda V)^{*,q} & \xrightarrow{\quad} & T^{*,q}(E_f) \\ \downarrow & & \downarrow \\ (T(X^{m-1}) \otimes_{f\#} \Lambda V)^{*,q} \oplus (T(\coprod \Delta[m]) \otimes_{f\#} \Lambda V)^{*,q} & \longrightarrow & T^{*,q}(E_f|_{X^{m-1}}) \oplus T^{*,q}(E_f|_{\coprod \Delta[m]}) \\ \downarrow & & \downarrow \\ (T(\coprod \partial \Delta[m]) \otimes_{f\#} \Lambda V)^{*,q} & \xrightarrow{\quad} & T^{*,q}(E_f|_{\coprod \partial \Delta[m]}) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

The right-hand column is exact because $T^{*,q}$ is a cohomology theory for $q \geq 1$. Unfortunately, this is not quite true for the left-hand column: indeed, for any simplicial set Y , we have a direct sum decomposition of graded \mathbb{Q}_q -modules

$$(6.5) \quad (T(Y) \otimes (\Lambda V))^{*,q} = \bigoplus_{n \geq 0, n\bar{k} \leq q} T^{*,q-n\bar{k}}(Y) \otimes (\Lambda^{(n)} V)^{*,n\bar{k}} \otimes \mathbb{Q}_q$$

as in the proof of Lemma 6.2. Furthermore, if $q - n\bar{k} \geq 1$ we have a short exact Mayer-Vietoris sequence

$$0 \rightarrow T^{*,q-n\bar{k}}(X) \rightarrow T^{*,q-n\bar{k}}(X^{m-1}) \oplus T^{*,q-n\bar{k}}(\coprod \Delta[m]) \rightarrow T^{*,q-n\bar{k}}(\coprod \partial \Delta[m]) \rightarrow 0,$$

which stays exact after tensoring with $(\Lambda^{(n)} V)^{*,n\bar{k}}$. However, for $q = n\bar{k}$ we need to be careful: the sequence

$$0 \rightarrow T^{*,0}(X) \rightarrow T^{*,0}(X^{m-1}) \oplus T^{*,0}(\coprod \Delta[m]) \rightarrow T^{*,0}(\coprod \partial \Delta[m])$$

is indeed exact for all $m \geq 1$, but the final map is surjective, only if $m \geq 2$, in general. For any $q' \geq 0$ let us therefore consider the cochain complex

$$\bar{T}^{*,q'}(\coprod \partial \Delta[m]) := \text{im} \left((T^{*,q'}(X^{m-1}) \oplus T^{*,q'}(\coprod \Delta[m])) \rightarrow T^{*,q'}(\coprod \partial \Delta[m]) \right),$$

which coincides with $T^{*,q'}(\coprod \partial \Delta[m])$ if $q' \geq 1$ or if $m \geq 2$. For any prime $l > q$ we get an inclusion of cochain complexes

$$\left(\bar{T}(\coprod \partial \Delta[m]) \otimes_{f\#} \Lambda V \right)^{*,q} \otimes \mathbb{F}_l \hookrightarrow \left(T(\coprod \partial \Delta[m]) \otimes_{f\#} \Lambda V \right)^{*,q} \otimes \mathbb{F}_l,$$

which is a $(t-1)$ -equivalence for $t \geq 1$ and $q \geq \bar{t}$ and is a t -equivalence for $t \geq k$ and $q \geq \max\{\bar{t}, \bar{k} + 1\}$. This is implied by the direct sum decomposition (6.5) and an argument similar to the one for the case $q = n\bar{k}$ in the proof of Lemma 6.2. Furthermore, the left hand column in Diagram (6.4) becomes exact after replacing $T(\coprod \partial \Delta[m]) \otimes_{f\#} \Lambda V$ by $\bar{T}(\coprod \partial \Delta[m]) \otimes_{f\#} \Lambda V$.

After these preliminaries a five lemma argument together with the induction hypothesis proves that $\Gamma_f : T(X) \otimes_{f\#} \Lambda(V) \rightarrow T(E_f)$ is a primary weak equivalence satisfying condition k^+ , thus finishing the induction step.

Finally, we assume that X is not of finite dimension. In this case we observe that the restriction maps $(T^{*,q}(X) \otimes_{f\#} \Lambda V) \otimes \mathbb{F}_l \rightarrow (T^{*,q}(X^m) \otimes_{f\#} \Lambda V) \otimes \mathbb{F}_l$ and $T^{*,q}(E_f) \otimes \mathbb{F}_l \rightarrow T^{*,q}(E_f|_{X^m}) \otimes \mathbb{F}_l$ induce isomorphisms in cohomology up to degree $m - 1$ for any prime $l > q$.

As a first application of the tame Hirsch lemma we study Eilenberg-MacLane complexes. The following result says that the assumption on the unit $\Psi_{\Lambda V}$ upon which the above proof of the tame Hirsch lemma was based always holds.

Proposition 6.6. *Let V be an elementary complex of type (d, \bar{k}) with $d \geq k \geq 1$. Then the unit*

$$\Psi_{\Lambda V} : \Lambda V \rightarrow T(\|\Lambda V\|)$$

is a primary weak equivalence satisfying condition k^+ .

The proof starts with the following lemma, which formulates one of the guiding principles of tame homotopy theory.

Lemma 6.7. *Let $d, q \geq 1$ and let l be a prime with $l > q$. Let $\Lambda(v)$ be the free (unfiltered) graded commutative \mathbb{F}_l -algebra generated by a variable v of degree d . Then the algebra map*

$$\Lambda(v) \rightarrow H^*(K(\mathbb{Q}_q, d); \mathbb{F}_l)$$

induced by sending v to the canonical generator of $H^d(K(\mathbb{Q}_q, d); \mathbb{F}_l)$ is an isomorphism in degree less than or equal to $d + 2l - 3$ and is an isomorphism in all degrees, if $d = 1$.

Proof. The result is clear, if $d = 1$, because then both sides are concentrated in degree 1. For an arbitrary prime l , a classical result of Cartan and Serre says that $H^*(K(\mathbb{Z}, d); \mathbb{F}_l)$ is a free commutative graded algebra over \mathbb{F}_l in one generator of degree d and further generators of degrees at least $d + 2(l - 1)$ as the first reduced Steenrod power operation for the prime l raises degree by $2(l - 1)$.

The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}_q$ induces an isomorphism $H^*(K(\mathbb{Q}_q, d); \mathbb{F}_l) \rightarrow H^*(K(\mathbb{Z}, d); \mathbb{F}_l)$, because $l > q$. This finishes the proof of Lemma 6.7. \square

Proposition 6.8. *Proposition 6.6 holds if V is concentrated in degree d and $V^{d, \bar{k}} \cong \mathbb{Q}_{\bar{k}}$.*

Proof. We start by noting that for each $q' \geq \bar{k}$ and for any prime $l > q'$, the unit $\Psi_{\Lambda V}$ induces an isomorphism

$$(6.9) \quad H^d((\Lambda V)^{*,q'} \otimes \mathbb{F}_l) \rightarrow H^d(T^{*,q'}(\|\Lambda V\|) \otimes \mathbb{F}_l)$$

by Proposition 4.5 and Theorem 3.5.

We first treat the case $d = 1$ separately. In this case we have $k = 1$. Let $q \geq 1$ and $l > q$. Then $q \geq \bar{k}$ and the cohomology $H^*(\Lambda(V)^{*,q} \otimes \mathbb{F}_l)$ is concentrated in degree 1 and isomorphic to \mathbb{F}_l . The same holds for $H^*(T^{*,q}(\|\Lambda(V)\|) \otimes \mathbb{F}_l)$ by the Cenkl-Porter theorem. Hence, $\Psi_{\Lambda V}$ is a primary weak equivalence satisfying condition k^+ .

From now on, we assume $d \geq 2$. Let $t \geq 1$ and $q \geq \bar{t}$. We distinguish the cases $t < k$ and $t \geq k$. In the first case we have $t < d$ (recall $k \leq d$) and we will show that $\Psi_{\Lambda V} : (\Lambda V)^{*,q} \otimes \mathbb{F}_l \rightarrow T^{*,q}(\|\Lambda V\|) \otimes \mathbb{F}_l$ is a t -equivalence for $l > q$. That the induced map in cohomology is an isomorphism up to degree t is trivial, because $t < d$. The injectivity in degree $t + 1$ is again trivial, if $q < \bar{k}$, because $(\Lambda V)^{t+1,q} = 0$ in this case. For $q \geq \bar{k}$ this injectivity follows from the isomorphism (6.9).

It remains to study the case $t \geq k$. Let $l > q$ be a prime. We will show that $\Psi_{\Lambda V} : (\Lambda V)^{*,q} \otimes \mathbb{F}_l \rightarrow T^{*,q}(\|\Lambda V\|) \otimes \mathbb{F}_l$ is a $(t + 1)$ -equivalence. This will finish the proof of Proposition 6.8.

It follows from the Cenkl-Porter theorem and from Lemma 6.7 that for all $\bar{k} \leq q' \leq q$ the graded groups $H^*(T^{*,q'}(\|\Lambda(V)\|) \otimes \mathbb{F}_l)$ and $\Lambda(v) \otimes \mathbb{F}_l$ are canonically isomorphic up to degree $d + 2l - 3$, where v denotes a variable of degree d . Furthermore, these isomorphisms are compatible with multiplications on filtration levels q'_1 and q'_2 with $\bar{k} \leq q'_1, q'_2 \leq q$ and $q'_1 + q'_2 \leq q$. Note that the assumptions $t \geq k$ and $l > q \geq \bar{t}$ imply $l > \bar{k}$ so that Lemma 6.7 can be applied. As $d \geq 2$ and $l \geq 2$, we obtain the inequality $d + 2l - 3 \geq l + 1$. And because $l > q \geq \bar{t} \geq t$, the isomorphism $H^*(T^{*,q'}(\|\Lambda(V)\|) \otimes \mathbb{F}_l) \cong \Lambda(v) \otimes \mathbb{F}_l$ holds up to degree $t + 2$. For a proof of the assertion that $\Psi_{\Lambda V}$ is a $(t + 1)$ -equivalence, it remains to show that for all $n \geq 1$ with $nd \leq t + 1$ we get the inequality $n\bar{k} \leq q$. For then we can apply induction on n starting with the isomorphism (6.9) with $q' = \bar{k}$. Notice that in degree $t + 2$, we need not care about the case $n\bar{k} > q$, because then $(\Lambda V)^{t+2,q} \otimes \mathbb{F}_l = 0$ so that the map in question is certainly injective. For $n = 1$ the required inequality holds, because $\bar{k} \leq \bar{t} \leq q$. For $n \geq 2$ it follows from the chain of inequalities $n\bar{k} \leq n\bar{d} \leq \bar{t} \leq q$ where the second inequality uses the admissibility of the filtration function and the assumptions $d, n \geq 2$.

This finishes the proof of Proposition 6.8. \square

We can now assemble the proof of Proposition 6.6.

First, let V be concentrated in degree d . We proceed by induction on the rank of $V^{d,\bar{k}}$. If this rank is equal to 1, we apply Proposition 6.8. Now let $V^{d,\bar{k}}$ be a free $\mathbb{Q}_{\bar{k}}$ -module of rank $n + 1$ with generators v_1, \dots, v_{n+1} . Applying the tame Hirsch lemma and Proposition 6.8, the map

$$T(\|\Lambda(v_1, \dots, v_n)\|) \otimes \Lambda(v_{n+1}) \rightarrow T(\|\Lambda(v_1, \dots, v_n)\| \times \|\Lambda(v_{n+1})\|) = T(\|\Lambda V\|)$$

is a primary weak equivalence satisfying condition k^+ . Here $\Lambda(v_{n+1})$ is the free filtered CGDA generated by v_{n+1} . Hence we need to show that the map

$$\Lambda(v_1, \dots, v_n) \otimes \Lambda(v_{n+1}) \rightarrow T(\|\Lambda(v_1, \dots, v_n)\|) \otimes \Lambda(v_{n+1})$$

is a primary weak equivalence satisfying condition k^+ . But this follows from Lemma 6.3 and the induction hypothesis. The assumption on filtration level 0 holds, because $\|\Lambda(v_1, \dots, v_n)\|$ is connected.

We are now able to prove the general case of Proposition 6.6. Let (V, η) be an elementary complex of type (d, \bar{k}) with $d \geq k \geq 1$. We can split V into two elementary complexes V_0 of type (d, \bar{k}) and V_1 of type $(d + 1, \bar{k})$ which are concentrated in degrees d and $d + 1$, respectively. The differential η in V is then regarded as a map of filtered cochain complexes $\eta : \Sigma V_0 \rightarrow V_1$. Applying the simplicial realization functor yields a simplicial map

$$\|\Lambda(\eta)\| : \|\Lambda V_1\| \rightarrow \|\Lambda(\Sigma V_0)\|.$$

Because the unit $\Psi_{\Lambda V_0}$ is a primary weak equivalence satisfying condition k^+ , which has already been shown, the tame Hirsch lemma applies and we get a primary weak equivalence satisfying condition k^+

$$\Gamma_{\|\Lambda(\eta)\|} : T(\|\Lambda V_1\|) \otimes_{\Psi_{\Lambda V_1} \circ \eta} \Lambda V_0 \rightarrow T(E_{\|\Lambda(\eta)\|}).$$

By an argument similar to the one used in the proof of Proposition 4.4, we see that $E_{\|\Lambda(\eta)\|} = \|\Lambda V\|$ and by Lemma 6.3, the map

$$\Lambda V = \Lambda V_1 \otimes_{\eta} \Lambda V_0 \rightarrow T(\|\Lambda V_1\|) \otimes_{\Psi_{\Lambda V_1} \circ \eta} \Lambda V_0$$

is a primary weak equivalence satisfying condition k^+ . This finishes the proof of Proposition 6.6.

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